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NOTE.

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J. HAMBLIN SMITH.

CAMBRIDGE, *April* 1880.

KEY TO

ELEMENTARY GEOMETRY.

Page 12.

Exercise 1. Let F be the other point in which the circles intersect.
Draw the straight lines AF, BF.
Then $\because$ A is the centre of ⊙ BFD,
$\therefore$ $AF=AB$;
and $\because$ B is the centre of ⊙ AFE,
$\therefore$ $BF=AB$.
$\therefore$ $AF=BF$, and ABF is an equilateral $\triangle$.

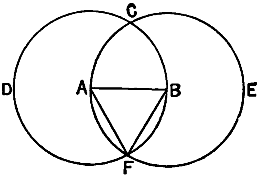

Fig. 1.

Ex. 2. Let AB, CD be two given straight lines, of which AB is the less.

Draw the straight line $AE=CD$, and with centre A and distance AE describe ⊙ EFH.

Produce AB to meet the ○ce in H.

Then $AH=AE$, and $\therefore$ $AH=CD$.

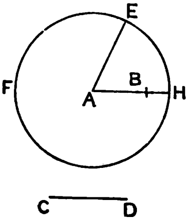

Fig. 2.

Ex. 3. With centre B and distance BC describe ⊙ CDE, and from B draw any line BD to meet the ○ce in D.

Then $BD=BC$.

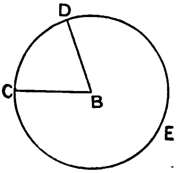

Fig. 3.

Ex. 4. Let *AB* be the given straight line on which the isosceles triangle is to be described, and let *CD* be the given straight line to which the equal sides of the triangle are to be equal.

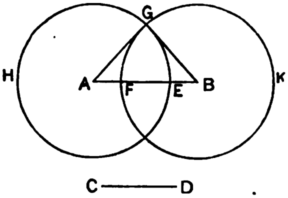

FIG. 4.

In *AB*, or *AB* produced, take $AE=CD$.

In *BA*, or *BA* produced, take $BF=CD$.

With centre *A* and distance *AE* describe the ⊙ *EGH*.

With centre *B* and distance *BF* describe the ⊙ *FGK*.

Draw the straight lines *AG*, *BG*.

$$\text{Then } \because AG=AE, \therefore AG=CD;$$
$$\text{and } \because BG=BF, \therefore BG=CD;$$
$$\therefore AGB \text{ is a } \triangle \text{ described as was required.}$$

Page 20.

EXERCISE 1. Taking the diagram of PROP. IX.

$$\because AE=AD, \therefore \angle ADE = \angle AED;$$
$$\text{and } \because FD=FE, \therefore \angle FED = \angle FDE;$$
$$\therefore \angle ADE \text{ with } \angle FDE = \angle AED \text{ with } \angle FED;$$
$$\therefore \angle ADF = \angle AEF.$$
$$\text{Then } \because AD=AE, \text{ and } FD=FE, \text{ and } \angle ADF = \angle AEF,$$
$$\therefore \angle EAF = \angle DAF.$$

Ex. 2. If the vertex *F* fall *within* the triangle *DAE*, or *without* the triangle *DAE*, the proof given in the proposition will hold good. But if the vertex *F* fall *on A*, the construction will fail.

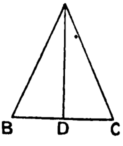

FIG. 5.

Page 21.

EXERCISE 1. In the △ *ABC*, let $AB=AC$.

Let *AD* bisect ∠ *BAC* and meet *BC* in *D*.

$$\text{Then } \because AB=AC, \text{ and } AD \text{ is common,}$$
$$\text{and } \angle BAD = \angle CAD;$$
$$\therefore BD=CD.$$

Ex. 2. In the diagram to Ex. 1, suppose that AD bisects BC in D.

Then $\because AB = AC$, and AD is common, and $BD = CD$,

$$\therefore \angle BAD = \angle CAD.$$

Ex. 3. Let AB be the given straight line.

Bisect AB in C, and with centre B and distance BC describe the ⊙ CDE. Produce AB to meet the ○ce in E.

Then $\because BE = CB$,

$\therefore BE$ is one-third of AE.

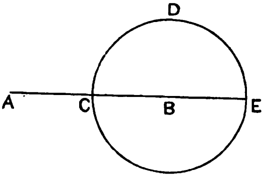

FIG. 6.

Page 22.

EXERCISE 1. Taking the diagram of PROP. IX., let O be the point in which AF and ED intersect.

Then $\because AD = AE$, and AO is common,

and $\angle DAO = \angle EAO$,

$\therefore DO = EO$, and $\angle AOD = \angle AOE$;

$\therefore AF$ bisects DE at right angles.

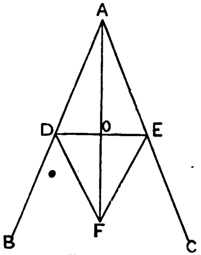

FIG. 7.

Ex. 2. Let DO, EO be two lines bisecting AB, AC, two sides of the equilateral △ ABC at right angles.

Join AO, BO, CO.

Then $\because AD = BD$, and OD is common,

and $\angle ADO = \angle BDO$,

$\therefore OA = OB$.

Similarly, it may be shown that $OA = OC$.

$\therefore OA$, OB, OC are all equal.

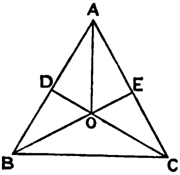

FIG. 8.

Ex. 3. In PROP. IX. we draw a straight line AF making equal angles with each of two straight lines BA, CA which meet.

In **Prop.** XI. we draw a straight line FC making equal angles with each of two straight lines CA, CB which meet, *and are in the same straight line.*

Page 23.

Exercise 1. Because the point C might be in such a position that it would be impossible to draw from it a line perpendicular to AB.

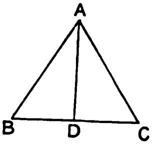

Fig. 9.

Ex. 2. In the $\triangle\, ABC$ let AD be a perpendicular on BC bisecting it.

Then $\because\ BD=CD$, and AD is common,

$$\text{and } \angle\, ADB = \angle\, ADC,$$

$$\therefore AB=AC.$$

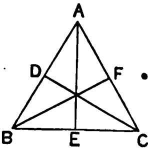

Fig. 10.

Ex. 3. Let D, E, F be the middle points of AB, BC, CA, the sides of an equilateral $\triangle$.

Then in $\triangle$s BAF, CBD,

$$\because\ BA=CB, \text{ and } AF=BD,$$

$$\text{and } \angle\, BAF = \angle\, CBD,$$

$$\therefore BF=CD.$$

Similarly, it may be shown that $BF=AE$.

Page 24.

Ex. 1.

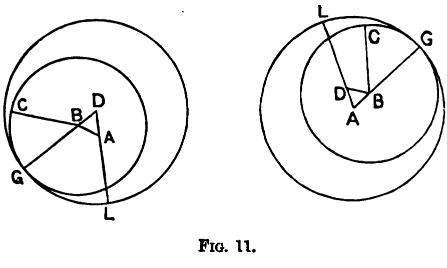

Fig. 11.

Ex. 2. Let BAC be the given angle.

Bisect $\angle BAC$ by the straight line AD.

Then bisect $\angle BAD$ by the straight line AE; and bisect $\angle CAD$ by the straight line AF.

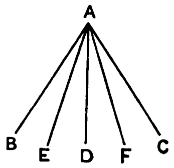

FIG. 12.

Ex. 3. Let ABC be the given isosceles triangle.

Then since $\angle DBC$ is half of $\angle ABC$,

and $\angle DCB$ is half of $\angle ACB$,

$\therefore \angle DBC = \angle DCB$, (Ax. 7.)

and $\therefore DB = DC$.

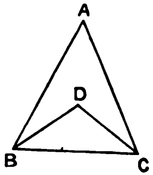

FIG. 13.

Ex. 4. In the $\triangle$s FBD, DCE,

$\because FB = DC$, and $BD = CE$,

and $\angle FBD = \angle DCE$,

$\therefore FD = ED$.

Similarly, it may be shown that $FD = FE$.

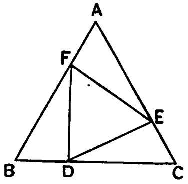

FIG. 14.

Ex. 5. Let AB be the given straight line, C and D the given points.

Join C, D by the straight line CD; and bisect CD in E.

Draw EF at right $\angle$s to CD, and let EF meet AB in F.

Join CF, DF.

Then $\because CE = DE$, and EF is common,

and $\angle CEF = \angle DEF$,

$\therefore CF = DF$.

Note.—If C and D be so situated that the line joining them is perpendicular to AB, the problem is impossible.

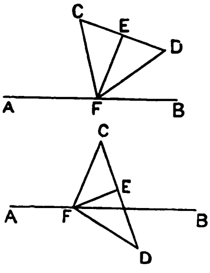

FIG. 15.

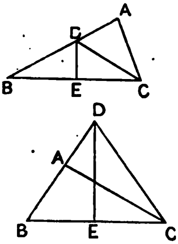

FIG. 16.

Ex. 6. Let ABC be the given $\triangle$, having the angle ABC an acute $\angle$.

Bisect BC in E, and draw $ED \perp$ to BC, meeting BA or BA produced in D. Join DC.

Then $\because BE = CE$, and ED is common,

and $\angle BED = \angle CED$,

$\therefore DB = DC$.

Ex. 7. In $\triangle$s AFC, AGB,

$\because FA = GA$, and $AC = AB$, and $\angle FAC = \angle GAB$,

$\therefore FC = GB$, and $\angle BFC = \angle CGB$.

Next, in $\triangle$s BFC, CGB,

$\because BF = CG$, and $FC = GB$, and $\angle BFC = \angle CGB$,

$\therefore \angle FBC = \angle GCB$, and $\angle BCF = \angle CBG$.

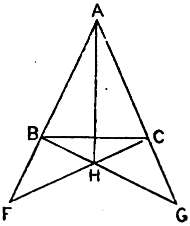

FIG. 17.

From $\angle FBC$ take $\angle CBG$, and from $\angle GCB$ take $\angle BCF$;

then $\angle FBH = \angle GCH$.

Then in $\triangle$s FBH, GCH,

$\because \angle BFH = \angle CGH$, and $\angle FBH = \angle GCH$,

and $FB = GC$,

$\therefore BH = CH$, and $FH = GH$. (I. B.)

Then in $\triangle$s AFH, AGH,

$\because AF = AG$, and AH is common, and $FH = GH$,

$\therefore \angle HAF = \angle HAG$.

Ex. 8. Let AD cut BC in E.

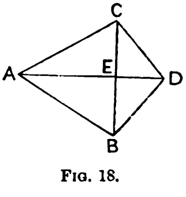

FIG. 18.

Then in $\triangle$s ABD, ACD,

$\because AB = AC$, AD is common, and $CD = BD$,

$\therefore \angle CAD = \angle BAD$.

Then in $\triangle$s ACE, ABE,

$\because CA = BA$, AE is common,

and $\angle CAE = \angle BAE$,

$\therefore CE = BE$, and $\angle AEC = \angle AEB$;

$\therefore AD$ bisects BC at right angles.

Ex. 9. To any point of the circumference *FEH*.

Ex. 10. Produce *AB*, *AC* two sides of $\triangle$ *ABC* to *D* and *E*, and let $\angle$ *DBC* = $\angle$ *ECB*.

In *CE* or *CE* produced take *CF*=*BD*, and join *DC*, *FB*, *FD*.

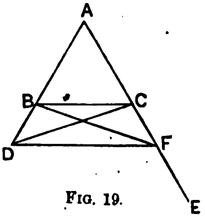

Fig. 19.

Then $\because$ *DB*=*FC*, and *BC* is common,

and $\angle$ *DBC* = $\angle$ *FCB*,

$\therefore$ *DC*=*FB*, and $\angle$ *BDC* = $\angle$ *CFB*,

and $\angle$ *BCD* = $\angle$ *CBF*.

From $\angle$ *CBD* take $\angle$ *CBF*, and from $\angle$ *BCF* take $\angle$ *BCD*,

then $\angle$ *FBD* = $\angle$ *DCF*.

Then $\because$ *FB*=*DC*, and *BD*=*CF*, and $\angle$ *FBD* = $\angle$ *DCF*,

$\therefore$ $\angle$ *BDF* = $\angle$ *CFD*, that is, $\angle$ *ADF* = $\angle$ *AFD* ;

$\therefore$ *AD*=*AF*, and $\therefore$ *AB*=*AC*.

Ex. 11. Since *AC*=*AD*, $\therefore$ $\angle$ *ADC* = $\angle$ *ACD*, and $\therefore$ $\angle$ *ADC* is greater than $\angle$ *BCD* ; much more is $\angle$ *BDC* greater than $\angle$ *BCD*.

Hence *BC* cannot be equal to *BD*, for then $\angle$ *BCD* would be equal to $\angle$ *BDC*.

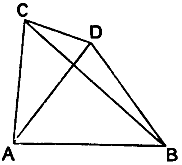

Fig. 20.

Ex. 12. In $\triangle$s *BAE*, *DAC*,

$\because$ *BA*=*DA*, and *AE*=*AC*,

and $\angle$ *BAE* = $\angle$ *DAC*,

$\therefore$ $\angle$ *BEA* = $\angle$ *DCA*=a rt. angle.

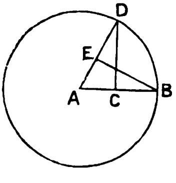

Fig. 21.

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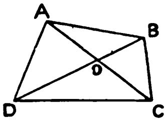

FIG. 22.

EXERCISE. Let $ABCD$ be the quadrilateral figure.
Then $\angle$s AOB, AOD together = two rt. $\angle$s,
and $\angle$s BOC, COD together = two rt. $\angle$s;
$\therefore$ the four $\angle$s at O together = four rt. $\angle$s.

Page 26.

EXERCISE. The words *upon the opposite sides of it* are necessary, because without them the words *the adjacent angles* would have no meaning.

Page 27.

EXERCISE 1. Let EF be the bisector of $\angle AED$, and EG be the bisector of $\angle BEC$.

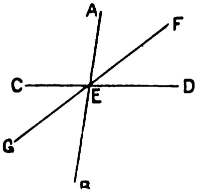

FIG. 23.

Then $\because \angle AED = \angle BEC$,
$\therefore \angle CEG = \angle FED$ (Ax. VII.)
$\therefore$ sum of $\angle$s AEG, AEF
= sum of $\angle$s AEC, CEG, AEF,
= sum of $\angle$s AEC, FED, AEF,
= sum of $\angle$s AEC, AED,
= two rt. $\angle$s;
$\therefore$ EF and EG are in the same st. line.

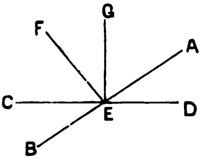

FIG. 24.

Ex. 2. Draw $EF \perp$ to AE, and $EG \perp$ to CE.
Then sum of $\angle$s GEF, GEA = a rt. $\angle$,
= sum of $\angle$s AED, GEA;
$\therefore \angle GEF = \angle AED$.

Ex. 3. In $\triangle$s AED, BEC,

$\because AE=BE$, and $ED=EC$,

and $\angle AED=\angle BEC$,

$\therefore$ the $\triangle$s are equal in all respects.

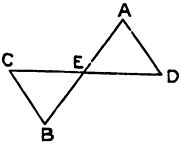

Fig. 25.

Page 29.

Exercise 1. If it be possible let AB, AC, AD be three equal st. lines drawn from A to meet the st. line MN.

Then $\because AB=AD$,

$\therefore \angle ABD=ADB$.

Now $\angle ACD$ is greater than $\angle ABD$;

$\therefore \angle ACD$ is greater than $\angle ADB$, that is, $\angle ADC$;

$\therefore AD$ is greater than AC, which is contrary to the hypothesis.

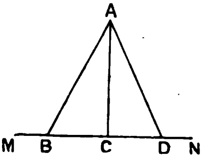

Fig. 26.

Ex. 2. Let FC make with BD the $\angle$s FCB, FCD, of which FCB is obtuse and FCD acute.

From F draw $FE\perp$ to BD: FE must fall on the side of the acute angle.

For if it fell otherwise, as FG, then would $\angle FCE$, an acute angle, be greater than the interior opposite $\angle FGC$, a right angle, which is absurd.

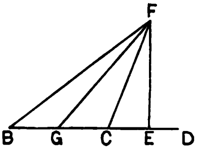

Fig. 27.

Page 31.

Exercise. In the $\triangle ABD$ let $\angle ABD=\angle ADB$. Then must $AB=AD$. For if not, let AB be greater than AD. Then will $\angle ADB$ be greater than $\angle ABD$, which is contrary to the hypothesis. Similarly, it may be shown that AB is not less than AD.

$\therefore AB=AD$.

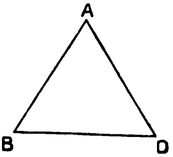

Fig. 28.

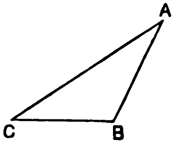

FIG. 29.

Page 32.

EXERCISE 1. In the $\triangle ABC$ let $\angle ABC$ be obtuse. Then each of the other angles must be acute, and $\therefore AC$ must be greater than either of the other sides.

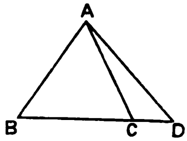

FIG. 30.

Ex. 2. Since $\angle ACB$ is greater than $\angle ADC$,
$\therefore \angle ABC$ is greater than $\angle ADC$,
that is, $\angle ABD$ is greater than $\angle ADB$,
and $\therefore AD$ is greater than AB.

Ex. 3. Draw AB, AC, AD from A to meet the straight line EF, and let AB be $\perp$ to EF, and let AC be nearer than AD to AB.

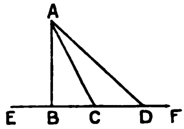

FIG. 31.

Then since $\angle ABC$ is a rt. angle,
$\therefore \angle ACB$ is an acute angle,
$\therefore AC$ is greater than AB.
Also, since $\angle ACD$ is greater than $\angle ABC$,
$\therefore \angle ACD$ is an obtuse angle,
and $\therefore \angle ADC$ is an acute angle,
and $\therefore AD$ is greater than AC.

Page 33.

EXERCISE 1. Let $ABCD$ be a quadrilateral; and join AC.

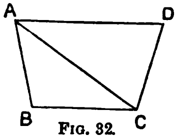

FIG. 32.

Then $\because AB$, BC together are greater than AC, $\therefore AB$, BC, CD together are greater than AC, CD together.

But AC, CD together are greater than AD;
$\therefore AB$, BC, CD together are greater than AD.

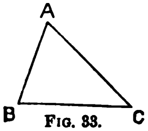

FIG. 33.

Ex. 2. Let ABC be any triangle.
Then AB, AC together are greater than BC.
Take from each AB.
Then AC is greater than the difference between BC and AB.

Ex. 3. Let O be any pt. within, or outside, the quadrilateral $ABCD$.

Then ∵ OA, OB are together greater than AB,
and OB, OC „ than BC,
and OC, OD „ than CD,
and OD, OA „ than DA;
∴ twice the sum of OA, OB, OC, OD is greater than sum of AB, BC, CD, DA;
∴ sum of OA, OB, OC, OD is greater than half the sum of AB, BC, CD, DA.

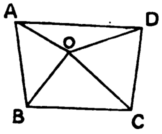

FIG. 34.

Ex. 4. Bisect BC, a side of the △ABC in D, join AD, and produce it to E, making $DE = DA$.

Join EC.

Then ∵ $BD = CD$, and $DA = DE$, and ∠BDA = ∠CDE,

∴ $AB = CE$.

Now sum of AC, CE is greater than AE,
∴ sum of AC, AB is greater than AE, that is, than twice AD.

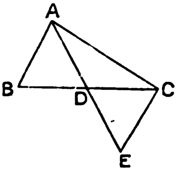

FIG. 35.

Page 34.

EXERCISE 1. Produce AD to meet BC in F, and join DB.

Then sum of AC, CF is greater than AF.
∴ sum of AC, CF, FB is greater than sum of AF, FB,
∴ sum of AC, CB is greater than sum of AD, DF, FB.
But sum of DF, FB is greater than sum of DE, EB;
∴ sum of AC, CB is greater than sum of AD, DE, EB.

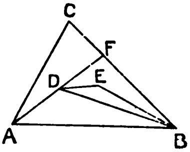

FIG. 36.

Ex. 2. Let O be any point within the △ABC. Then—
∵ sum of AB, BC is greater than sum of OA, OC,
and sum of BC, CA „ than sum of OB, OA,
and sum of CA, AB „ than sum of OB, OC;
∴ twice sum of AB, BC, CA is greater than twice sum of OA, OB, OC;
∴ sum of AB, BC, CA is greater than sum of OA, OB, OC.

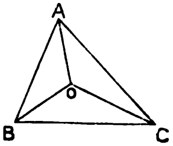

FIG. 37.

Next, $\because$ sum of OA, OB is greater than AB,
and sum of OB, OC „ than BC,
and sum of OC, OA „ than CA;
$\therefore$ twice sum of OA, OB, OC is greater than sum of AB, BC, CA;
and $\therefore$ sum of OA, OB, OC is greater than half the sum of AB, BC, CA.

Page 35.

Exercise. Take A and C two equal st. lines, and take B a st. line equal to one-half of either A or C, and then proceed as in the Proposition.

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Miscellaneous Exercises.

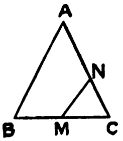

Fig. 38.

1. Difference between MB and MN is equal to difference between MC and MN, and is therefore less than CN,

that is, less than difference between AC, AN,
that is, less than difference between AB, AN.

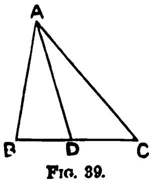

Fig. 39.

2. Since $\angle ADB$ is greater than $\angle CAD$,
$\therefore \angle ADB$ is greater than $\angle BAD$;
$\therefore AB$ is greater than BD.
Again, since $\angle ADC$ is greater than $\angle BAD$,
$\therefore \angle ADC$ is greater than $\angle DAC$;
$\therefore AC$ is greater than CD.

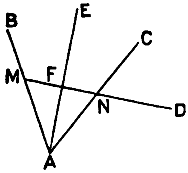

Fig. 40.

3. Draw AE bisecting $\angle BAC$.

Draw $DF \perp$ to AE, meeting AB and AC in M and N, and meeting AE in F.

Then $\angle FAM = \angle FAN$, and $\angle AFM = \angle AFN$, and AF is common to the $\triangle$s AFM, AFN,

$\therefore AM = AN$.

4. The construction is the same as in Ex. 3, and then $\angle FMA = \angle FNA$.

5. Draw AD bisecting $\angle BAC$, and AE bisecting $\angle BAG$.

Then $\because$ sum of $\angle$s BAG, BAC = two rt. $\angle$s,
and $\angle BAE$ = half of $\angle BAG$,
and $\angle BAD$ = half of $\angle BAC$,
$\therefore$ sum of $\angle$s BAE, BAD = a rt. $\angle$,
that is, $\angle EAD$ is a rt. $\angle$.

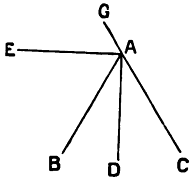

FIG. 41.

6. Let ABC be a triangle, having side AC greater than AB.

Let AD, the bisector of $\angle BAC$, meet BC in D, and let AE bisect BC in E.

From AC cut off $AF = AB$, and join FD.

Then $\because AB = AF$, and AD is common,
and $\angle BAD = \angle FAD$,
$\therefore \angle ADF = \angle ADB$,
and $\therefore \angle ADE$ is greater than $\angle ADB$,
$\therefore \angle ADE$ is greater than $\angle AED$,
$\therefore AE$ is greater than AD.

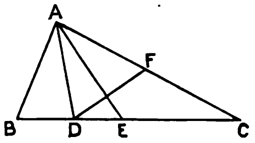

FIG. 42.

7. From any one of the angular points of the $\triangle ABC$, as A, draw AD to meet BC in D.

Then $\angle ADC$ is greater than $\angle ABD$, and $\angle ADB$ is greater than $\angle ACD$;

$\therefore$ sum of $\angle$s ADC, ADB is greater than sum of $\angle$s ABD, ACD;

$\therefore$ sum of $\angle$s ABD, ACD is less than two rt. $\angle$s,
that is, sum of $\angle$s ABC, ACB is less than two rt. $\angle$s.

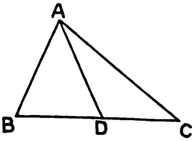

FIG. 43.

8. Since $AB = AD$, and AE is common, and $\angle BAE = \angle DAE$;

$\therefore \angle ABE = \angle ADE$.

But $\angle ADE$ is greater than $\angle ACB$;
$\therefore \angle ABE$ is greater than $\angle ACB$,
that is, $\angle ABC$ is greater than $\angle ACB$.

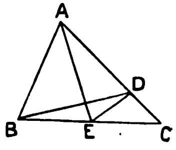

FIG. 44.

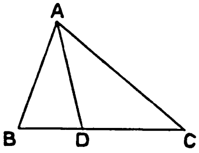

FIG. 45.

9. Let AD be the bisector of $\angle BAC$.

Then, as in Ex. 2, we can show that AB is greater than BD, and that AC is greater than CD;

$\therefore AB, AC$ together are greater than BC.

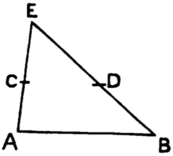

FIG. 46.

10. Let AB be the given side.

At A make $\angle CAB$ = one of the given $\angle$s.

At B make $\angle DBA$ = the other given $\angle$;

AC and BD being drawn so as to lie on the same side of AB.

Then AC, BD will, if produced, meet in some pt. E. (POST. 6.)

$\therefore ABE$ will be the required $\triangle$.

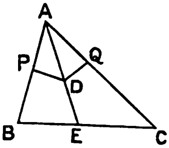

FIG. 47.

11. Let D be any pt. in AE, the bisector of $\angle BAC$.

Draw $DP, DQ \perp$ to AB, AC.

Then $\because \angle DAP = \angle DAQ$, and $\angle DPA = \angle DQA$, and AD is common to the $\triangle$s ADP, ADQ,

$\therefore DP = DQ$.

Page 49.

1. Through E, a point equidistant from the parallel lines AB, CD, draw $FEG \perp$ to AB and CD. Then $FE = GE$.

Draw HEK meeting AB in H and CD in K.

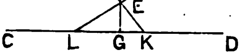

FIG. 48.

Draw LEM meeting CB in L and AB in M.

Then $\because \angle ELG = \angle EMF$, and $\angle LGE = \angle MFE$, and $FE = GE$,

$\therefore FM = LG$.

And $\because \angle EHF = \angle EKG$, and $\angle EFH = \angle EGK$, and $FE = GE$,

$\therefore HF = GK$.

$\therefore$ sum of FM, HF = sum of LG, GK.

$\therefore HM = LK$.

2. Let AD, the bisector of $\angle BAC$, meet BC in D.

Draw DE parallel to AB, meeting AC in E, and DF parallel to AC, meeting AB in F.

Then $\because \angle ADE = \angle DAF$,

and $\angle ADF = \angle DAE$,

$\therefore \angle ADE = \angle ADF$.

Then $\because \angle ADE = \angle ADF$,

and $\angle DAE = \angle DAF$, and AD is common,

$\therefore DE = DF$.

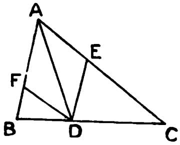

Fig. 49.

3. Let EF, joining the parallel lines AB, CD, be bisected in O and through O draw MON to meet AB in M, and CD in N.

Then $\because \angle MEO = \angle NFO$,

and $\angle EMO = \angle ONF$, and $OE = OF$;

$\therefore MO = NO$.

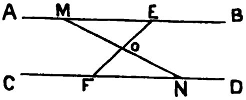

Fig. 50.

Page 50.

Exercise 1. Let AB, AC be parallel to DE, DF respectively, and place them so that AC cuts DE in O.

Then $\because AB$ is $\parallel$ to DE,

$\therefore \angle BAC = \angle AOD$;

and $\because AC$ is $\parallel$ to DF,

$\therefore \angle AOD = \angle EDF$;

$\therefore \angle BAC = \angle EDF$.

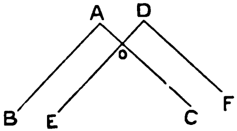

Fig. 51.

Ex. 2. Let AB be $\perp$ to CD.

Join AD and draw DE $\perp$ to AD.

Produce CD to F.

Then shall $\angle BAD = \angle EDF$.

For $\angle ADF =$ sum of $\angle$s ABD, DAB;

$\therefore$ sum of $\angle$s ADE, $EDF =$ sum of $\angle$s ABD, DAB;

Also, $\angle ADE =$ a rt. $\angle = \angle ABD$;

$\therefore \angle EDF = \angle DAB$.

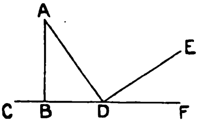

Fig. 52.

Page 51.

Ex. 1. Let A be the given pt., BC the given st. line, E the given $\angle$.

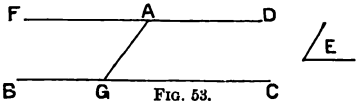

FIG. 53.

Through A draw $FD \parallel$ to BC; at A make $\angle\, FAG = \angle\, E$, and let AG meet BC in G. Then $\angle\, AGC = \angle\, FAG = \angle\, E$.

Ex. 2. Let DE, FG be the two $\parallel$ st. lines, and MN the given line. In FG take $OP = MN$, and with centre O and distance OP descr. a ⊙ cutting DE in R. Join OR, and draw through A a st. line $ABC \parallel$ to OR, meeting DE in B and FG in C, and join OB.

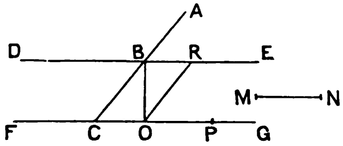

FIG. 54.

Then $\because \angle\, BCO = \angle\, DBC$, and $\angle\, DBC = \angle\, BRO$;

$\therefore \angle\, BCO = \angle\, BRO$, and $\angle\, CBO = \angle\, BOR$, and OB is common;

$\therefore BC = RO = MN$.

Page 52.

EXERCISE 1. Since any one of the angles is less than a right angle, the sum of the other two must be greater than a right angle, and therefore the sum of any two is greater than the third.

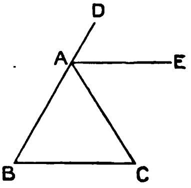

FIG. 55.

Ex. 2. Let ABC be an isosceles $\triangle$, having $AB = AC$.

Produce BA to D, and draw AE bisecting $\angle\, CAD$.

Then $\angle\, CAD =$ sum of $\angle$ s ABC, ACB,
$=$ twice $\angle\, ACB$;
$\therefore \angle\, EAC = \angle\, ACB$,
and $\therefore AE$ is parallel to BC.

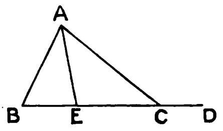

FIG. 56.

Ex. 3. Sum of $\angle$s ABD, ACD,
$=$ sum of $\angle$s ABD, AED, CAE,
$=$ sum of $\angle$s ABD, AED, BAE,
$=$ sum of $\angle$s AED, AED
$=$ twice $\angle\, AED$.

Ex. 4. Let ABC be an isosceles $\triangle$, having $AB=AC$.

Produce BC to D, and let OB, OC, the bisectors of the equal $\angle$s, meet in O.

Then $\angle\ COB$ with $\angle$s $OBC, OCB=$ two rt. $\angle$s;

$\therefore\ \angle\ COB$ with $\angle\ ACB=$two rt. $\angle$s.

But $\angle\ ACD$ with $\angle\ ACB=$two rt. $\angle$s;

$\therefore\ \angle\ COB=\angle\ ACD$.

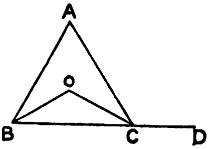

FIG. 57.

Ex. 5. Let ABC be a $\triangle$. Produce BA to D, and let AE, the bisector of $\angle\ CAD$, be $\parallel$ to BC.

Then $\because\ \angle\ DAE=$interior $\angle\ ABC$,

and $\angle\ EAC=$alternate $\angle\ ACB$,

$\therefore\ \angle\ ABC=\angle\ ACB$,

and $\therefore\ AB=AC$.

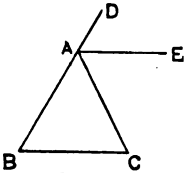

FIG. 58.

Page 54.

EXERCISE 1. Let $ABCD$ be any quadrilateral, and produce AB to E, BC to F, CD to G, DA to H.

Then, by Cor. II., sum of exterior $\angle$s $=$ four rt. $\angle$s,

and, by Cor. I., sum of interior $\angle$s $=$ four rt. $\angle$s.

$\therefore$ sum of exterior $\angle$s$=$sum of interior $\angle$s.

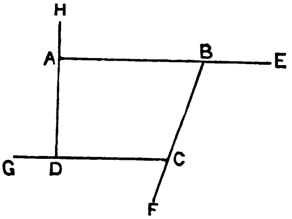

FIG. 59.

Ex. 2. Sum of the six angles with four rt. $\angle$s$=$twelve rt. $\angle$s,

$\therefore$ sum of the six angles$=$eight rt. $\angle$s.

Ex. 3. Sum of the five equal angles with four rt. $\angle$s$=$ten rt. $\angle$s,

$\therefore$ sum of the five equal $\angle$s$=$six rt. $\angle$s,

$\therefore$ each angle$=\frac{6}{5}$ of a rt. $\angle$.

Ex. 4. Sum of intr. $\angle$s with four rt. $\angle$s$=$twice as many rt. $\angle$s as the figure has sides.

But sum of intr. $\angle$s in the figure$=$eight rt. $\angle$s;

$\therefore$ twelve rt. $\angle$s$=$twice as many rt. $\angle$s as the figure has sides;

$\therefore$ the figure has six sides.

Ex. 5. Let n represent the number of sides.
Then $\frac{7}{4}\times n+4=2n$, or, $7n+16=8n$;
$\therefore\ n=16$.

Page 56.

1. Let ABC be a $\triangle$, and produce BA, BC, and let OA, OC, the bisectors of the exterior $\angle$s, meet in O. Draw OD, OE, $OF\perp$s to BA, BC, CA, or to these produced.

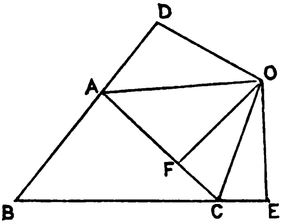

Fig. 60.

Then $\because\ \angle DAO=\angle FAO$, and $\angle ADO=\angle AFO$, and AO is common;
$\therefore\ OD=OF$;
and $\because\ \angle OCE=\angle OCF$, and $\angle OEC=\angle OFC$, and OC is common;
$\therefore\ OF=OE$.

2. Let BAC be a right angle.

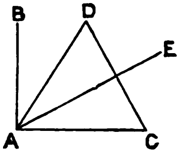

Fig. 61.

On AC describe the equilat. $\triangle ADC$, on the side of AC on which AB stands, and bisect $\angle DAC$ by the st. line AE.

Then $\because\ \angle DAC=\frac{1}{3}$ of two rt. $\angle$s,
$\therefore\ \angle EAC=\frac{1}{3}$ of a rt. $\angle$,
$\therefore\ \angle EAD=\frac{1}{3}$ of a rt. $\angle$,
and $\therefore\ \angle DAB=\frac{1}{3}$ of a rt. $\angle$.

3. Draw BO, CO the bisectors of $\angle$s ABC, ACB meeting in O. Draw OD, OE, $OF\perp$ to AB, BC, CA, and join AO.

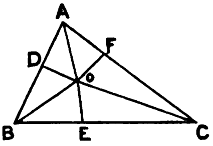

Fig. 62.

Then we have to show that OA bisects $\angle BAC$.

Now $\because\ \angle OBE=\angle OBD$,
and $\angle OEB=\angle ODB$, and OB is common,
$\therefore\ OD=OE$;
and $\because\ \angle OCF=\angle OCE$, and
$\angle OFC=\angle OEC$, and OC is common,
$\therefore\ OF=OE$, and $\therefore\ OF=OD$.

Then $\because$ $OF=OD$, and AO is common, and $\angle$s OFA, ODA are rt. $\angle$s,

$\therefore \angle OAD = \angle OAF$. (See Cor. to Prop. E, page 43).

4. Let OD, OE, bisecting AB, BC at rt. $\angle$s meet in O.

Join AO, BO, CO, and draw OF to F the middle point of AC. We have then to show that OF is $\perp$ to AC.

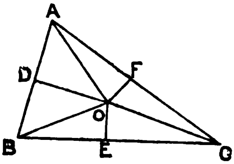

Fig. 63.

Now $\because$ $BE=CE$, and OE is common, and $\angle OEB = \angle OEC$,

$\therefore OB=OC$,

and $\because$ $BD=AD$, and OD is common, and $\angle ODB = \angle ODA$,

$\therefore OB=OA$, and $\therefore OA=OC$.

Then $\because$ $OA=OC$, and OF is common, and $FA=FC$,

$\therefore \angle OFA = \angle OFC$, and $\therefore OF$ is $\perp$ to AC.

5. In $\triangle ABC$ draw $AD \perp$ to BC, and AE bisecting $\angle BAC$.

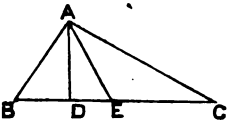

Fig. 64.

Then sum of $\angle$s ABC, $BAE = \angle AEC$,

$=$sum of $\angle$s DAE, ADE,

$=$sum of $\angle$s DAE, ADB,

$=$sum of $\angle$s DAE, DAC, ACB,

$=$sum of $\angle$s DAE, DAE, EAC, ACB.

Now $\angle BAE = \angle EAC$,

$\therefore \angle ABC=$twice $\angle DAE$ with $\angle ACB$;

$\therefore$ difference between $\angle ABC$ and $\angle ACB=$twice $\angle DAE$.

6. $\because \angle BDA = \angle EDA$,

and $\angle BAD = \angle EAD$, and AD is common,

$\therefore BD=ED$.

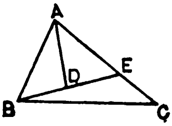

Fig. 65.

7. Let ABC be a rt.-angled $\triangle$, having $\angle BAC$ a rt. $\angle$.

Make $\angle BAD = \angle ABC$;

then $\angle DAC = \angle ACB$.

Then $\because \angle BAD = \angle ABD$, $\therefore AD=DB$;

and $\because \angle DAC = \angle ACD$, $\therefore AD=DC$.

$\therefore ABD$ and ACD are isosceles triangles.

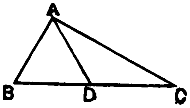

Fig. 66.

8. Produce BA, DC to meet in F.
Draw $EM \parallel$ to CD, meeting FB in M.
In MB make $MG = MF$.
Join GE and produce it to meet FD in H.

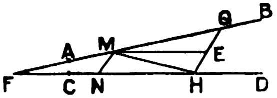

FIG. 67.

Draw $MN \parallel$ to GH, and join MH.
Then $\because \angle GME = \angle MFN$,
and $\angle MGE = \angle FMN$, and $MG = FM$,
$\therefore MN = GE$.
And $\because \angle EMH = \angle MHN$,
and $\angle EHM = \angle NMH$, and MH is common,
$\therefore EH = MN$;
and $\therefore EH = GE$.

9. Let R be the middle pt. of PQ.

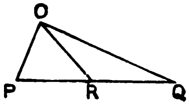

FIG. 68.

(1.) If $OR = PR$, then $\angle ROP = \angle OPR$,
and $OR = QR$, and $\angle ROQ = \angle OQR$;
$\therefore \angle POQ =$ sum of $\angle$s OPQ, OQP;
$\therefore \angle POQ$ is a rt. angle.

(2.) If OR is greater than PR, then $\angle ROP$ is less than $\angle OPR$, and OR is greater than QR, and $\angle ROQ$ is less than $\angle OQR$;
$\therefore \angle POQ$ is less than the sum of $\angle$s OPQ, OQP;
$\therefore \angle POQ$ is an acute angle.

(3.) If OR is less than PR, then $\angle ROP$ is greater than $\angle OPR$, and OR is less than QR, and $\angle ROQ$ is greater than $\angle OQR$;
$\therefore \angle POQ$ is greater than the sum of $\angle$s OPQ, OQP;
$\therefore \angle POQ$ is an obtuse angle.

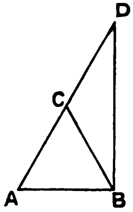

FIG. 69.

10. Let AB be the given st. line.

Describe on AB an equilateral triangle ACB, and produce AC to D, making $CD = AC$.

Join DB, this line shall be $\perp$ to AB.

For BC, the line bisecting AD, is equal to the half of AD;

$\therefore ABD$ is a rt. angle, by Ex. 9.

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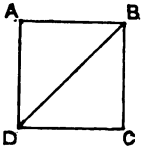

FIG. 70.

1. Let $ABCD$ be a square, and BD a diagonal.
Then $\because AB = AD$, $\therefore \angle ABD = \angle ADB$, and since sum of $\angle$s ABD, ADB, BAD = two rt. $\angle$s, and $\angle BAD$ is a rt. $\angle$,

$\therefore$ each of the $\angle$s ABD, ADB is half a rt. $\angle$.

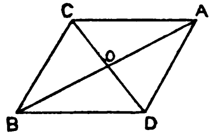

FIG. 71.

2. Let AB, CD bisect each other in O.
Join AD, DB, BC, CA.

Then $\because AO = BO$, and $CO = DO$,
and $\angle AOC = \angle BOD$,
$\therefore \angle OAC = \angle OBD$,
$\therefore AC$ is $\parallel$ to BD.

In the same way it may be shown that BC is $\parallel$ to AD,
$\therefore ADBC$ is a ▱.

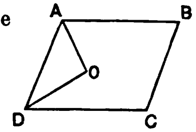

FIG. 72.

3. Let $ABCD$ be a ▱, and let AO, DO, the bisectors of $\angle$s BAD, CDA meet in O.

Then $\because$ sum of $\angle$s BAD, CDA = two rt. $\angle$s,
$\therefore$ sum of $\angle$s OAD, ODA = a rt. $\angle$,
$\therefore \angle AOD$ is a rt. $\angle$.

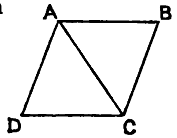

FIG. 73.

4. Let $ABCD$ be a ▱, and let AC bisect each of the $\angle$s BAD, DCB.

Then $\because \angle BAC = \angle CAD$,
and $\angle BAC = \angle ACD$,
$\therefore \angle CAD = \angle ACD$, and $\therefore AD = CD$.

Therefore all the sides of the ▱ are equal.

5. Let $ABCD$ be a quadrilateral, having $\angle ABC = \angle ADC$, and $\angle BAD = \angle DCB$.

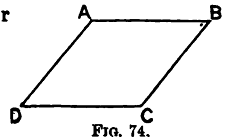

FIG. 74.

Then since the four angles together = four rt. $\angle$s,
$\therefore$ sum of $\angle$s BAD, ADC = two rt. $\angle$s.
and sum of $\angle$s ABC, BAD = two rt. $\angle$ s,
$\therefore AB$ is $\parallel$ to CD, and AD is $\parallel$ to BC.

6. Let $ABCD$ be a quadrilateral, and let $AB=CD$, and $AD=BC$.

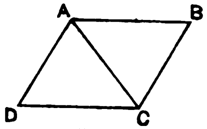

FIG. 75.

Join AC.

Then $\because AB=CD$, and $BC=AD$,

and AC is common,

$\therefore \angle BCA = \angle DAC$, and $\angle BAC = \angle ACD$;

$\therefore AD$ is $\parallel$ to BC, and AB is $\parallel$ to CD.

7. Let $ABCD$ be a rhombus, and let $\angle BAD$ be two-thirds of two rt. angles.

Then $\angle BCD$ = two-thirds of two rt. angles. Join AC.

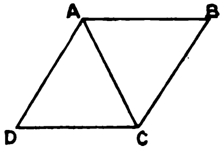

FIG. 76.

Then $\because BA=DA$, and AC is common,

and $BC=DC$,

$\therefore \angle BAC = \angle DAC$, and $\angle ACB = \angle ACD$.

Hence $\angle BAC$ = one-third of two rt. angles,

and $\angle ACB$ = one-third of two rt. angles,

$\therefore \angle ABC$ = one-third of two rt. angles,

and $\triangle ABC$ is equilateral.

Similarly it may be shown that $\triangle ADC$ is equilateral.

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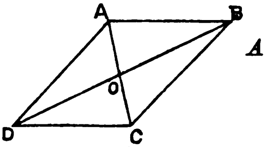

FIG. 77.

EXERCISE 1. Let the diagonals of the ▱ $ABCD$ meet in O.

Then $\because \angle ABO = \angle CDO$,

and $\angle BAO = \angle DCO$, and $AB=CD$,

$\therefore AO=CO$, and $BO=DO$.

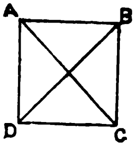

FIG. 78.

Ex. 2. Let $ABCD$ be a rectangle.

Then in the $\triangle$s ABC, DCB,

$\because AB=DC$, and BC is common,

and $\angle ABC = \angle DCB$,

$\therefore AC=BD$.

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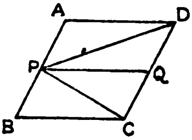

FIG. 79.

Exercise 1. Draw $PQ \parallel$ to AD and BC.
Then $\because APQD$ is a ▱, $\therefore \triangle PAD = \triangle PQD$;
and $\because PBCQ$ is a ▱, $\therefore \triangle PBC = \triangle PQC$;
$\therefore$ sum of $\triangle$s $PAD, PBC = \triangle PDC$.

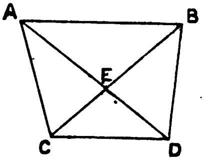

FIG. 80.

Ex. 2. $\triangle ACD = \triangle BCD$, because they are on the same base CD, and between the same parallels.

Take from each $\triangle CED$.

Then $\triangle AEC = \triangle BED$.

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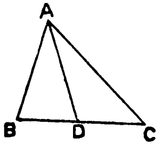

FIG. 81.

Exercise 1. Let AD bisect BC in D.
Then $\triangle ABD = \triangle ADC$, because they are on equal bases and between the same parallels, a line through A being assumed parallel to BC.

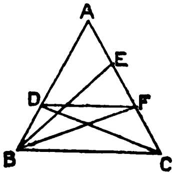

FIG. 82.

Ex. 2. In CA make $CF = BD$ or AE.
Join BF, DF.
Then $\because DB = FC$, and BC is common,
and $\angle DBC = \angle FCB$,
$\therefore \triangle DBC = \triangle FCB$.
Now $\triangle FCB = \triangle ABE$, (I. 38.)
$\therefore \triangle DBC = \triangle ABE$.

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Exercise 1. Join CD, and let it meet EP in O.
Then since $\triangle PEB = \triangle ABC$, take from each $\triangle BEC$;

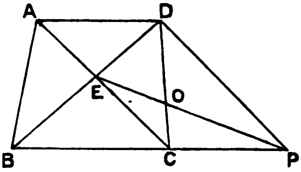

Fig. 83.

$\therefore \triangle EPC = \triangle ABE.$

Also, since $\triangle ABD = \triangle ACD.$ (I. 37.)

take from each $\triangle AED$;

$\therefore \triangle ABE = \triangle EDC.$

$\therefore \triangle EPC = \triangle EDC.$

Take from each $\triangle EOC$;

$\therefore \triangle OCP = \triangle EOD.$

Add to each $\triangle DOP$;

$\therefore \triangle PCD = \triangle PED.$

$\therefore AC$ is || to PD.

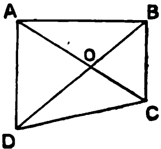

Fig. 84.

Ex. 2. Let the diagonals of the quadrilateral $ABCD$ intersect in O, and let $\triangle AOB = \triangle DOC$.
Add to each $\triangle BOC$.
Then $\triangle ABC = \triangle BDC$,
and $\therefore AD$ is || to BC.

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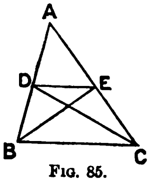

Fig. 85.

Exercise 1. Let D and E be the middle pts. of AB, AC. Join DE.

Then $\triangle CDE = \triangle ADE$, (I. 38.)

and $\triangle BDE = \triangle ADE$; (I. 38.)

$\therefore \triangle CDE = \triangle BDE.$

$\therefore DE$ is || to BC. (I. 39.)

Ex. 2. Let D, E, F be the middle pts. of AB, BC, CA.
Join DE, EF, FD.

Then DF is $\parallel$ to BC,
and FE is $\parallel$ to AB,
$\therefore DBEF$ is a ▱ ;
$\therefore \triangle DFE = \triangle DBE$.

Now $\triangle ADF = \triangle BDE$, $\because FE$ is $\parallel$ to AB ;
and $\triangle FEC = \triangle DBE$, $\because DF$ is $\parallel$ to BC ;
$\therefore$ the four triangles are equal.

Also note that $DF = BE =$ the half of BC.

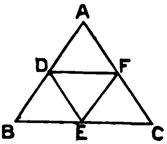

FIG. 86.

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EXERCISE 1. Let $ABCD$ be the ▱, O the pt. without. Join OA, OB, OC, OD. Through O draw $EOF \parallel$ to AB, meeting DA, CB produced in E and F.

Then $\triangle ODC =$ half of ▱ $EDCF$,
and $\triangle OAB =$ half of ▱ $EABF$,
$\therefore$ difference between $\triangle ODC$ and $\triangle OAB =$ half of ▱ $ABCD$.

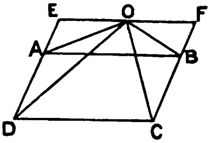

FIG. 87.

Ex. 2. Take O, any pt. within the ▱ $ABCD$, and join OA, OB, OC, OD.

Through O draw $EOF \parallel$ to AD.

Then $\triangle AOD =$ half of ▱ $ADFE$,
and $\triangle OBC =$ half of ▱ $EBCF$,
$\therefore$ sum of $\triangle$s AOD, $OBC =$ half of ▱ $ABCD$.

Similarly, sum of $\triangle$s AOB, $COD =$ half of ▱ $ABCD$.

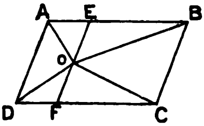

FIG. 88.

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EXERCISE 1. Let $ABCD$ be the given ▱, M the given $\angle$.
Produce AB to E making $BE = AB$.
Make $\angle FAE = \angle M$, and let AF meet DC or DC produced in F, and join FE, BF.

Then $\triangle AFE$ is double of $\triangle ABF$,
and ▱ $ABCD$ is double of $\triangle ABF$;
$\therefore \triangle AFE =$ ▱ $ABCD$, and has an $\angle FAE = \angle M$.

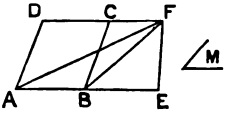

FIG. 89.

Ex. 2. Let ABC be the given $\triangle$.

Draw $AD \parallel$ to BC, bisect BC in O, and with centre O and distance equal to half the sum of BA, AC, describe a circle cutting AD in E, and join OE, and through C draw $CF \parallel$ to OE, meeting AD in F.

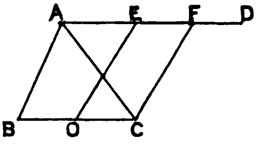

FIG. 90.

Then $EOCF$ is a ▱, the sum of whose sides is equal to the sum of the sides of $\triangle ABC$,

for sum of OE, CF = sum of AB, AC,

and sum of OC, $EF = BC$.

Also ▱ $EOCF = \triangle ABC$, since $OC = \frac{1}{2} \cdot BC$.

Ex. 3. Let ABC be an isosceles $\triangle$. Draw $AO \perp$ to BC, then AO bisects BC. Complete the rectangle $AOCD$.

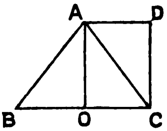

FIG. 91.

Now AC is greater than AO, $\because \angle AOC$ is a rt. $\angle$; and $\therefore AB$ is greater than CD.

Also, AD, OC together $= BC$.

$\therefore$ sum of AB, BC, CA is greater than sum of AO, OC, CD, DA.

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EXERCISE 1. If O be not in AC, let it lie on the side of AC nearest to B, and let the line drawn through $O \parallel$ to BC cut AC in P.

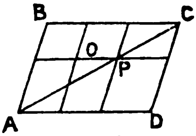

FIG. 92.

Through P draw another line $\parallel$ to CD.

Then ▱ $BP =$ ▱ PD; (I. 43.)

$\therefore$ ▱ OB is less than ▱ OD, which is contrary to the hypothesis.

Similarly it may be shown that O does not lie on the side of AC nearest to D, and $\therefore O$ will be in AC.

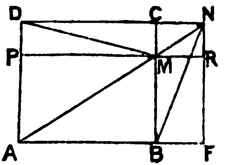

FIG. 92a.

Ex. 2. Complete the ▱ $CBFN$.

Draw through M a line $\parallel$ to DN, meeting AD in P, and FN in R.

Then ▱ $DM =$ ▱ MF. (I. 43.)

Now ▱ $DM =$ twice $\triangle MDC$; (I. 34.)

and ▱ $MF =$ twice $\triangle MBN$; (I. 41.)

$\therefore \triangle MBN = \triangle MDC$.

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Miscellaneous Exercises.

1. Let AC, a diagonal of the quadrilateral $ABCD$, bisect the other diagonal, BD, in O.

Then $\because BO=DO, \therefore \triangle AOB = \triangle AOD$;

and $\because BO=DO, \therefore \triangle COB = \triangle COD$;

$\therefore$ sum of $\triangle$s AOB, COB = sum of $\triangle$s AOD, COD;

$\therefore \triangle ABC = \triangle ADC$.

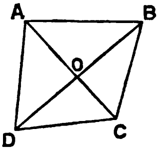

FIG. 93.

2. Let the diagonals of the ▱ $ABCD$ meet in M.

Take O any pt. in DB, or in DB produced, and join OA, OC.

Then $\because MA=MC$. (Ex. 1, p. 60.)

$\therefore \triangle MAB = \triangle MCB$,

and $\triangle MAO = \triangle MCO$;

$\therefore \triangle AOB = \triangle COB$.

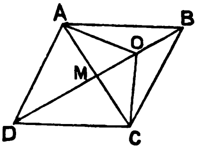

FIG. 94.

3. Let $ABDC$ be a trapezium, having $AB \parallel$ to CD.

Let M, N be the middle pts. of AB, CD, and join MN, AN, BN.

Then $\because AM=BM, \therefore \triangle AMN = \triangle BMN$;

and $\because CN=DN, \therefore \triangle ANC = \triangle BND$;

$\therefore$ sum of $\triangle$s AMN, ANC = sum of $\triangle$s BMN, BND.

$\therefore$ fig. $ACNM$ = fig. $BDNM$.

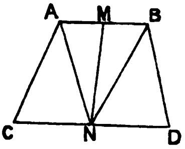

FIG. 95.

4. Sum of $\triangle$s CPD, BPC = fig. $DPBC$

= sum of $\triangle$s CDB, BPD.

Now $\triangle CDB = \frac{1}{2}$ ▱ $ABCD$,

$= \triangle ABC$,

= sum of $\triangle$s APB, BPC, APC;

$\therefore$ sum of $\triangle$s CPD, BPC = sum of $\triangle$s APB, BPC, APC, BPD;

$\therefore \triangle CPD$ = sum of $\triangle$s APB, APC, BPD;

$\therefore$ difference of $\triangle$s CPD, APB = sum of $\triangle$s APC, BPD.

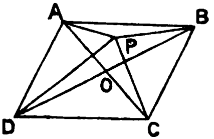

FIG. 96.

5. Let $ABCD$ be a ▱, and let $AC=AB$.

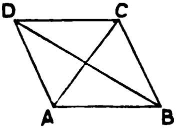

FIG. 97.

Now since each of the equal angles in an isosceles △ must be less than a rt. ∠, (I. 17.)

∴ ∠ ABC is less than a rt. ∠;

∴ ∠ BAD is greater than a rt. ∠. (I. 29.)

∴ BD is greater than AD or AB. (I. 19, Ex. 1.)

∴ BD is greater than any side of the figure.

6. Let $ABCD$ be a ▱.

Draw EF, FG, GH, HE through A, B, C, D ∥ to the diagonals of ▱ $ABCD$.

Then ∵ EF and HG are both ∥ to BD;

∴ EF is ∥ to HG.

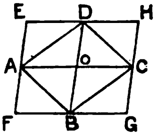

FIG. 98.

Similarly FG is ∥ to EH;

∴ $EFGH$ is a ▱.

Let O be the intersection of AC and BD.

Then ▱ $AOBF$ = twice △ AOB,

▱ $BOCG$ = twice △ BOC,

▱ $CHDO$ = twice △ DOC,

▱ $DOAE$ = twice △ DOA;

∴ ▱ $EFGH$ = twice ▱ $ABCD$.

7. Let ABC, DEF be two △s having $AB=DE$, $AC=DF$, and ∠ BAC the supplement of ∠ FDE.

Complete the ▱ $ABCG$, and join AG.

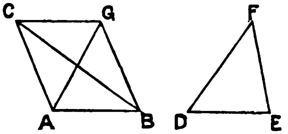

FIG. 99.

Then ∠ BAC is the supplement of ∠ ABG.

∴ ∠ ABG = ∠ FDE, and $AB=DE$, and $BG=FD$;

∴ △ ABG = △ EDF.

But △ ABC = △ ABG;

∴ △ ABC = △ DEF.

8. Let ABC be the given △, and P the given pt. in AC.

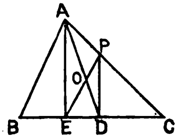

FIG. 100.

Bisect BC in D, join AD, PD, and draw AE ∥ to PD.

Join PE, cutting AD in O.

Then shall PE bisect the △ ABC.

For ∵ AE is ∥ to PD, ∴ △ APD = △ EPD; take from each △ POD; then △ AOP = △ EOD.

Also, $\because BD = CD$, $\therefore \triangle ABD = \triangle ACD$, of which the parts AOP, EOD are equal, and $\therefore$ fig. $ABEO =$ fig. $PODC$;

$\therefore$ sum of $ABEO$ and $\triangle AOP =$ sum of $PODC$ and $\triangle EOD$;

$\therefore$ fig. $ABEP = \triangle PEC$.

9. Since $\triangle AEC = \triangle ABE$ (I. 38.)

$= \triangle EBD$;

$\therefore \triangle ABC = \triangle ADE$.

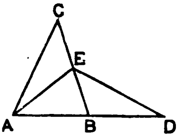

Fig. 101.

10. Take the diagram of I. 43, and join EG, HK: these lines shall be $\parallel$.

Join HE, KG.

Then $\because AF = FC$, $\therefore \triangle HEF = \triangle KFG$; add to each $\triangle HKF$.

Then $\triangle HEK = \triangle HGK$;

and $\therefore EG$ is $\parallel$ to HK.

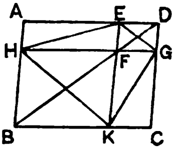

Fig. 102.

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Exercise 1. Let AB, CD be the two given lines.

Draw $EF = CD$, and from E draw $EG = AB$ and $\perp$ to EF. Complete the rectangle $EFHG$, which will be described as required.

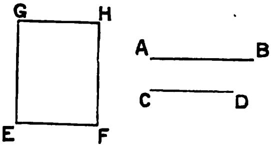

Fig. 103.

Ex. 2. Let $ABCD$, $EFGH$ be squares on equal st. lines AB, EF.

Apply $EFGH$ to $ABCD$, so that E lies on A and EF on AB, then $\because EF = AB$,

F will coincide with B.

And since $\angle DAB = \angle HEF$, EH will fall on AD, and since $EH = AD$, H will coincide with D. Similarly it may be shown that G will coincide with C.

$\therefore EFGH$ coincides with and is $\therefore$ equal to $ABCD$.

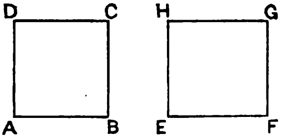

Fig. 104.

Ex. 3. Let $ABCD$, $EFGH$ be equal squares (diagram of Ex. 2). Apply $EFGH$ to $ABCD$, so that E coincides with A, and EF falls on AB and EH on AD; then must F coincide with B. For if F falls between A and B, then H falls between A and D, and G will fall inside $ABCD$, and $EFGH$ will be enclosed by $ABCD$, which is impossible. And if F falls on AB produced, then G will fall outside $ABCD$, and $ABCD$ will be enclosed by $EFGH$, which is impossible.

$\therefore$ F will coincide with B, and $\therefore$ $EF = AB$.

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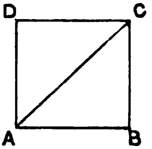

Fig. 105.

Exercise 1. Let $ABCD$ be the given square, AC a diagonal.

Then ABC is a right-angled $\triangle$,

and sq. on AC = sum of sqq. on AB, BC,

= twice sq. on AB,

= twice the given square.

Ex. 2. Let A, B, C be three given lines.

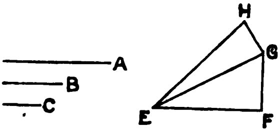

Fig. 106.

Take $EF = A$, and from F draw $FG = B$, and $\perp$ to EF.

Join EG. Then sq. on EG = sum of sqq. on A, B.

From G draw $GH = C$, and $\perp$ to EG. Join EH.

Then sq. on EH = sum of sqq. on EG, GH.

= sum of sqq. on A, B, C.

Ex. 3. Let ABC be a triangle, having $\angle ACB$ equal to the sum of the other two angles.

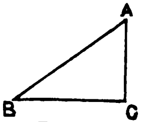

Fig. 107.

Then $\angle ACB$ is a rt. $\angle$. (I. 32.)

Also since the sum of the squares of 4 and 3 is 16 + 9, or 25, and since the square of 5 is 25, AB must contain five parts each equal to one of the equal parts into which BC and CA are divisible.

Ex. 4. Sum of sqq. on AC, BC = sq. on AB,
= sq. on DE,
= sum of sqq. on DF, EF.
But sq. on AC = sq. on DF;
$\therefore$ sq. on BC = sq. on EF;
$\therefore$ $BC = EF$, and $\therefore$ the triangles are equal in all respects.

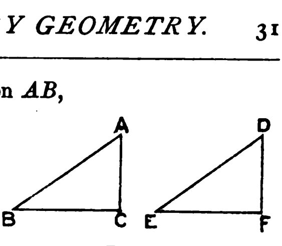

FIG. 108.

Ex. 5. Let AB be the given st. line. Draw $AC \perp$ to AB, and make $AC = AB$. Join CB. Bisect $\angle BCA$ by CD meeting AB in D, and draw $DE \perp$ to BC.

Then $\because$ $\angle ACD = \angle ECD$, and $\angle CAD = \angle CED$, and CD is common,

$\therefore AD = DE$.

Also, since $\angle EBD$ = half a rt. $\angle$ = $\angle EDB$,

$\therefore EB = DE$.

Then sq. on DB = sum of sqq. on DE, BE,
= twice sq. on DE,
= twice sq. on DA.

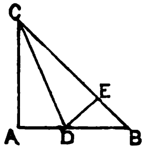

FIG. 109.

Ex. 6. Let ABC be a right-angled $\triangle$, having $\angle ABC$ a rt. $\angle$.
Draw AD to meet BC in D.
Then sum of sqq. on BC, AD
= sum of sqq. on BC, AB, BD,
= sum of sqq. on AC, BD.

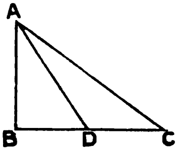

FIG. 110.

Ex. 7. Let ABC be any triangle, and let AD be drawn at rt. angles to BC.
Then $\because$ sq. on AC = sum of sqq. on CD, AD,
and sq. on AB = sum of sqq. on BD, AD;
difference between sqq. on AC, AB
= difference between sqq. on CD, BD.

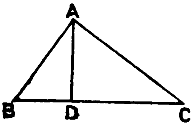

FIG. 111.

END OF BOOK I.

BOOK II.

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Exercise. Let *AB* be divided into any number of parts in *E*, *F*, *G*; and let *AC* be divided into any number of parts in *H*, *K*.

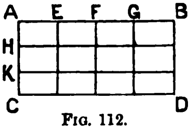

Fig. 112.

Place *AC* to make a rt. ∠ with *AB*.

Through *E*, *F*, *G*, *B* draw lines ‖ to *AC*.

Through *H*, *K*, *C* draw lines ‖ to *AB*.

Then the proof is the same as that in Prop. I.

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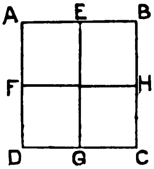

Fig. 113.

On *AB* describe a square *ABCD*.

Bisect *AB*, *AD* in the pts. *E* and *F*.

Draw *EG* ‖ to *AD*, and *FH* ‖ to *AB*.

Then the square *ABCD* is divided into four squares, each of which is equal to the square on *AE*.

∴ sq. on *AB* = four times sq. on *AE*.

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Exercise. Let *ABC* be the △, *BAC* the rt. ∠.

Draw *AD* ⊥ to *BC*.

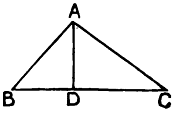

Fig. 114.

Then sq. on *BD* with sq. on *CD* with twice rect. *BD*, *DC* = sq. on *BC* (II. 4.)

= sq. on *BA* with sq. on *CA*,

= sum of sqq. on *BD*, *DA*, *DA*, *CD*;

∴ twice rect. *BD*, *DC* = twice sq. on *DA*;

∴ rect. *BD*, *DC* = sq. on *DA*.

Page 84.

EXERCISE. In PROP. V. AD is the sum of AC, CD,
DB is the difference of AC, CD;
$\therefore$ rect. AD, DB = difference of sqq. on AC, CD,
or, rect. AD, DB with sq. on CD = sq. on AC.

In PROP. VI. AD is the sum of CD, AC;
$\therefore$ AD is the sum of CD, CB,
DB is the difference of CD, CB;
$\therefore$ rect. AD, DB = difference of sqq. on CD, CB;
or, rect. AD, DB with sq. on CB = sq. on CD.

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EXERCISE. Since $\angle CGB$ = half a rt. $\angle$,
and $\angle CGH$ = a rt. $\angle$,
and $\angle HGD$ = half a rt. $\angle$,
$\therefore$ sum of $\angle$s CGB, CGH, HGD = two rt. $\angle$s;
$\therefore$ BGD is a st. line.

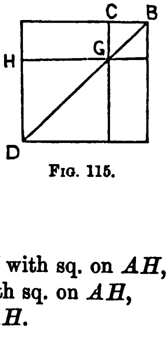

FIG. 115.

Page 89.

EXERCISE. By II. 7
sq. on AB with sq. on BH = twice rect. AB, BH with sq. on AH,
= twice sq. on AH with sq. on AH,
= three times sq. on AH.

Page 90.

EXERCISE. Let $ABCD$ be a trapezium, having $AB \parallel$ to CD.
Draw AE, BF $\perp$ to CD, or CD produced.

Then sq. on AC = sq. on AD with sq. on CD, with twice rect. CD, DE; and sq. on BD = sq. on BC with sq. on CD, with twice rect. CD, CF,

$\therefore$ sqq. on AC, BD = sqq. on AD, BC with twice sq. on CD with twice rect. CD, DE, with twice rect. CD, CF.

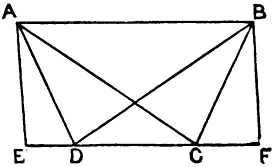

FIG. 116.

Again, rect. *AB*, *CD*=rect. *EF*, *CD*=rect. *CD*, *DE*, with rect. *CD*, *CD* with rect. *CD*, *CF*. (II. 1.)

∴ twice rect. *AB*, *CD*=twice rect. *CD*, *DE* with twice sq. on *CD* with twice rect. *CD*, *CF*;

∴ sqq. on *AC*, *BD*=sqq. on *AD*, *BC* with twice rect. *AB*, *CD*.

Note.—The angles at *C* and *D* have been drawn as *obtuse* angles. If either or both be *acute* angles, the proof is similar, but it will depend on Prop. XIII. with, or instead of, Prop. XII.

Page 91.

Exercise. Let *ABC* be any △, *AE* the ⊥ from *A* on *BC*, *AD* the line drawn from *A* to bisect *BC* in *D*.

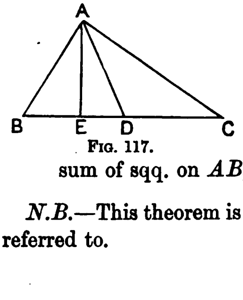

Fig. 117.

Then sq. on *AB*=sqq. on *BD*, *AD* diminished by twice rect. *BD*, *DE*; and sq. on *AC*=sqq. on *CD*, *AD* increased by twice rect. *CD*, *DE*;

∴, observing that *BD*=*CD*,

sum of sqq. on *AB*, *AC*=twice sum of sqq. on *BD*, *AD*.

N.B.—This theorem is of great importance, and it will be frequently referred to.

Page 93.

Miscellaneous Exercises on Book II.

1. Let ∠ *BAC* be a rt. ∠, and let *AD* be ⊥ to *BC*.

Then, since ∠ *ABC* is an acute ∠, sq. on *AC* with twice rect. *BC*, *BD*=sqq. on *AB*, *BC* (II. 13.)

=sqq. on *AB*, *AB*, *AC*;

∴ twice rect. *BC*, *BD*=twice sq. on *AB*;

∴ rect. *BC*, *BD*=sq. on *AB*.

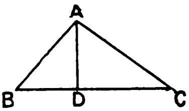

Fig. 118.

Similarly it may be shown that rect. *BC*, *CD* =sq. on *AC*.

2. Let $ABCD$ be a ▱, and draw AE, BF ⊥ to DC or DC produced. Then $DE = CF$.

Now sq. on AC = sqq. on AD, DC diminished by twice rect. CD, DE, and sq. on BD = sqq. on BC, DC increased by twice rect. CD, CF;

∴, observing that $AB = DC$,

sum of sqq. on AC, BD = sum of sqq. on AB, BC, CD, DA.

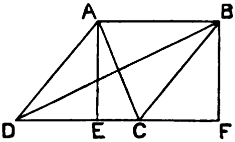

FIG. 119.

3. Let $ABCD$ be a rectangle, and let the diagonals intersect in P. Then they bisect each other, and AP, BP, CP, DP are all equal.

(I. 34, Ex. 1 and 2.)

Join OP. Then, as is proved in the Ex. to II. 13, sum of sqq. on AO, OC = twice sum of sqq. on AP, OP; and sum of sqq. on OB, OD = twice sum of sqq. on DP, OP.

∴, since $AP = DP$,

sum of sqq. on AO, OC = sum of sqq. on OB, OD.

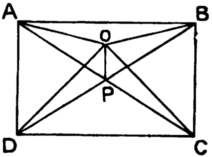

FIG. 120.

4. In the ▱ $ABCD$, let $BD = DC = AB$.

Now sum of sqq. on AC, DB
= sum of sqq. on AB, BC, CD, DA (Ex. 2.)
= sum of sqq. on DB, BC, DB, BC;
∴ sq. on AC = sq. on DB with twice sq. on BC;
∴ sq. on DB is less than sq. on AC by twice sq. on BC.

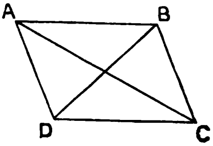

FIG. 121.

5. Let D be the side of the given square.

Draw $BE = D$ at rt. ∠s to AB.

With centre A and distance AE describe a ⊙, and let AB produced meet the ○ce in C.

Then rect. contained by the sum and difference of AB, AC,
= difference of sqq. on AC, AB, (II. B.)
= difference of sqq. on AE, AB,
= sq. on BE = sq. on D.

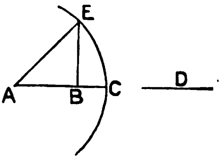

FIG. 122.

6. Let $ABCD$ be a quadrilateral, and let O, P be the middle pts. of its diagonals. Join OP, BP, DP.

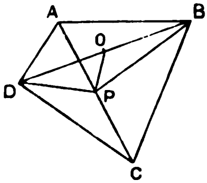

FIG. 123.

Then by Ex. to II. 13,

sum of sqq. on AB, BC = twice sum of sqq. on BP, CP;

and sum of sqq. on CD, DA = twice sum of sqq. on DP, CP;

∴ sum of sqq. on the four sides = twice sum of sqq. on BP, DP with four times sq. on CP.

But sum of sqq. on BP, DP = twice sum of sqq. on BO, OP; (II. 13, Ex.)

∴ sum of sqq. on the four sides = four times sum of sqq. on BO, OP, CP.

Also, sq. on AC = four times sq. on CP; (II. 2, Ex.)

and sq. on BD = four times sq. on BO; (II. 2, Ex.)

∴ sum of sqq. on the four sides = sum of sqq. on diagonals with four times sq. on OP.

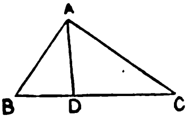

FIG. 124.

7. Let ABC be a △, and AD the ⊥ from A on BC, and let sq. on AD = rect. BD, DC.

Then sq. on BC = sum of sqq. on BD, DC with twice rect. BD, DC,

= sum of sqq. on BD, DC, DA, DA,

= sum of sqq. on BA, AC;

∴ ∠ BAC is a rt. angle. (I. 48.)

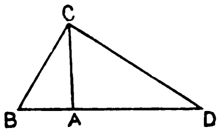

FIG. 125.

8. Let AB, AC be the given lines. Place them so as to be at right angles to each other. Join BC.

Draw CD ⊥ to BC, meeting BA produced in D.

Then, by Ex. to II. 4,

rect. BA, AD = sq. on AC.

9. Divide AB in any points C, D.

On AB describe the sq. $ABEF$.

In AF take AG = AC, and GH = CD, then HF = BD.

Divide AE into nine rectangles by drawing lines from C, D $\parallel$ to AF, and lines from G, H $\parallel$ to AB.

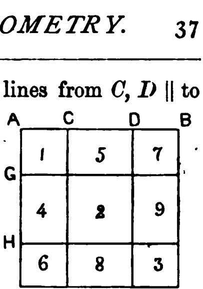

Fig. 126.

Then 1, 2, 3 are the squares on AC, CD, DB,
4, 5 are the rect. AC, CD,
6, 7 are the rect. AC, DB,
8, 9 are the rect. CD, DB;

∴ sq. on AB=sum of sqq. on AC, CD, DB with twice the rectangle contained by the parts, taken two and two.

10. Let $ABCD$ be a quadrilateral, and E, F, G, H the middle points of its sides.

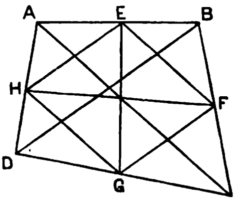

Fig. 127.

Then ∵ EF joins the middle pts. of AB, CB.

∴ EF is $\parallel$ to AC, and AC=twice EF.
(I. 40, Ex. 1.)

Similarly, HG is $\parallel$ to AC, and EH, FG are $\parallel$ to BD, and ∴ $EFGH$ is a ▱.

Now sq. on AC=four times sq. on EF.
(II. 2, Ex.)

∴ sum of sqq. on AC, BD=four times sum of sqq. on EF, FG,
=twice sum of sqq. on EF, FG, GH, HE,
=twice sum of sqq. on EG, FH.
(By Ex. 2.)

11. Let ABC be a △, D, E, F the middle points of BC, CA, AB.

Then sum of sqq. on AB, AC=twice sum of sqq. on AD, BD,
sum of sqq. on AC, CB=twice sum of sqq. on CF, BF,
sum of sqq. on CB, AB=twice sum of sqq. on BE, AE;

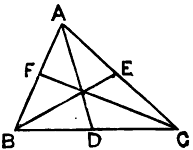

Fig. 128.

∴ twice sum of sqq. on AB, AC, CB=twice sum of sqq. on AD, CF, BE with twice sum of sqq. on BD, BF, AE;

∴ four times sum of sqq. on AB, AC, CB =four times sum of sqq. on AD, CF, BE with four times sum of sqq. on BD, BF, AE;

∴ four times sum of sqq. on AB, AC, CB=four times sum of sqq. on AD, CF, BE with sum of sqq. on CB, AB, AC;

∴ three times sum of sqq. on AB, AC, CB=four times sum of sqq. on AD, CF, BE.

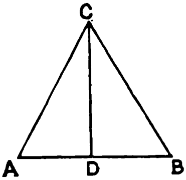

FIG. 129.

12. Sum of sqq. on CD, DA = sq. on CA,
= sq. on AB,
= sum of sqq. on AD, DB with twice rect. AD, DB;
$\therefore$ sq. on CD = sq. on DB with twice rect. AD, DB.

13. Since CD bisects AE,

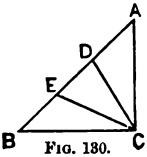

FIG. 130.

sum of sqq. on AC, CE = twice sum of sqq. on CD, DE;
and since CE bisects BD,
sum of sqq. on BC, CD = twice sum of sqq. on CE, DE.
$\therefore$ sum of sqq. on AC, BC, CE, CD = twice sum of sqq. on CD, CE, DE, DE;
$\therefore$, observing that sq. on AB = sum of sqq. on AC, BC, sq. on AB = sum of sqq. CD, CE, DE with three times sq. on DE.
Now since DE is one-third of AB, the sq. on DE is one-ninth of sq. on AB;
$\therefore$ two-thirds of sq. on AB = sum of sqq. on CD, CE, DE.

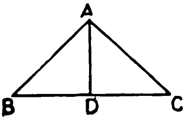

FIG. 131.

14. Let ABC be an isosceles right-angled $\triangle$, having $BA = AC$, and $\angle BAC$ the rt. $\angle$.
Draw $AD \perp$ to BC.
Then AD bisects $\angle BAC$ and also BC.
$\therefore \angle DAC = \angle ACD$,
and $\therefore DC = DA$.
$\therefore$ sq. on BC = four times sq. on DC = four times sq. on DA.

15. Let AB be the given st. line, and M the other line.

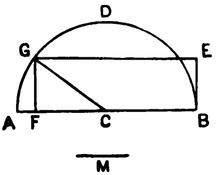

FIG. 132.

Bisect AB in C, and on AB describe the semicircle ADB.
Draw $BE = M$, and $\perp$ to AB; and draw $EG \parallel$ to BA to meet the semicircle in G; and draw $GF \parallel$ to EB.
Then rect. AF, FB with sq. on FC
= sq. on CB (II. 5.)
= sq. on CG,
= sum of sqq. on CF, FG.
$\therefore$ rect. AF, FB = sq. on FG,
= sq. on M.

Loci on page 104.

(1.) A circle described with the given pt. as the centre, and the given distance as the radius.

(2.) A straight line parallel to the given line.

(3.) A straight line parallel to the given line.

(4.) A straight line bisecting the angle.

(5.) A circle described with the centre of the given circle as its centre, and with a radius equal to the sum of the radius of the given circle and a straight line equal to the given distance.

(6.) Two straight lines bisecting the vertically opposite angles.

Page 116.

Miscellaneous Exercises on Books I. and II.

1. Let AB, CD intersect at right angles in O.
Then $\because$ $AO=OB$, and OC is common, and $\angle AOC = \angle BOC$,
$\therefore$ $AC=CB$; and similarly it may be shown that $AC=AD=DB=CB$.
Again $\because$ $OA=OC$, $\therefore$ $\angle OCA = \angle OAC$;
$\therefore$ $\angle OAC$ is half a rt. $\angle$.
Similarly, $\angle OAD$ is half a rt. $\angle$,
$\therefore$ $\angle CAD$ is a rt. angle; and $\therefore$ $ACBD$ is a square.

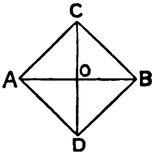

FIG. 133.

2. Let $ABCD$ be a ▱, and P a point in AB.
Bisect BD in F; join PF, and produce it to meet DC in E.
Then $\because$ $\angle FPB = \angle FED$,
and $\angle FBP = \angle FDE$, and $FB=FD$,
$\therefore$ $\triangle PFB = \triangle EFD$.
Also $\triangle ABD = \triangle BDC$.
$\therefore$, by subtraction, fig. $APFD$=fig. $BFEC$;
$\therefore$, adding the equal $\triangle$s FDE, FBP,
fig. $APED$=fig. $BPEC$.

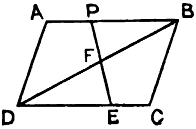

FIG. 134.

3. Produce DC to E, making $CE=DE$, and join EF.

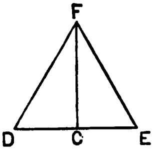

Fig. 135.

Then $\because DC=EC$, and CF is common,
and $\angle FCD = \angle FCE$,
$\therefore FD=FE$.
Now $\angle FDE$ is $\frac{2}{3}$ of a rt. $\angle$,
$\therefore \angle FED$ is $\frac{2}{3}$ of a rt. $\angle$,
and $\therefore \angle DFE$ is $\frac{2}{3}$ of a rt. $\angle$,
$\therefore FDE$ is an equilateral $\triangle$;
$\therefore FD=DE=$ twice DC.

4. Let AD, BE, CG the $\perp$s on the sides meet in F, which is proved on p. 56, Ex. 4.

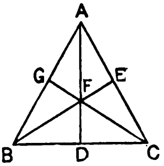

Fig. 136.

Then, since $\angle AFB$ is an obtuse $\angle$,
sq. on $AB=$ sum of sqq. on AF, FB with twice rect. AF, FD.
Now $AF=BF=CF$; and $BF=$ twice FD; (Ex. 3.)
$\therefore$ sq. on $AB=$ sum of sqq. on AF, AF with twice half the sq. on AF.
$=$ three times sq. on AF.

5. Let ABC be the given $\triangle$, and BC the side to which each side of the rhombus shall be equal.

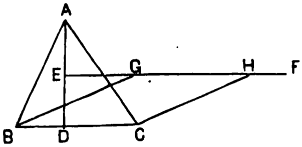

Fig. 137.

Draw $AD \perp$ to BC: bisect AD in E: draw $EF \parallel$ to BC, and with centre B and distance BC describe a ⊙ cutting EF in G.

Join GB and draw $CH \parallel$ to GB: then $GBCH$ is a rhombus, and area of $GBCH=$ rect. ED, $BC=\frac{1}{2}$ rect. AD, $BC=$ area of $\triangle ABC$.

Note.—The problem is impossible if BC be less than ED.

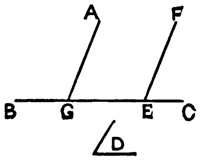

Fig. 138.

6. Let A be the given point, BC the given st. line, D the given angle.

Take E any point in BC, and at E make $\angle FEC= \angle D$. Through A draw $AG \parallel$ to FE, meeting BC in G.

Then $\angle AGC = \angle FEC = \angle D$.

7. Let A, B be the given pts.; DE the given line.

Draw $AD \perp$ to DE, and produce AD to C, making $DC = DA$. Join BC cutting DE in O. Join AO, and from any other pt. P in DE draw AP, BP. Then shall AO, OB be together less than AP, PB together.

For $\because AD = CD$, and OD is common,
and $\angle ADO = \angle CDO$,
$\therefore AO = CO$; and similarly $AP = CP$.
Also, $\angle AOD = \angle COD = \angle BOP$.
Now sum of AO, OB = sum of CO, OB
$= CB$;
and sum of AP, PB = sum of CP, PB,
which is greater than CB;
$\therefore$ sum of AP, PB is greater than sum of AO, OB.

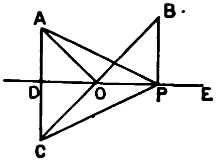

FIG. 139.

8. Let OP cut AB in M, and let OQ cut CD in N.

Then $\angle QOD$ is supplement of $\angle QOB$.
Now the diagonals of a $\square$ bisect each other,
and $\therefore DO = OB$, and $MB = \frac{1}{2} AB = \frac{1}{2} DC = DN$.
Hence in $\triangle$s DON, MOB,
$\because DO = BO$, and $MB = DN$, and $\angle MBO = \angle ODN$,
$\therefore \angle MOB = \angle DON$; that is, $\angle POB = \angle QOD$;
$\therefore \angle POB$ is the supplement of $\angle QOB$, and $\therefore POQ$ is a st. line.

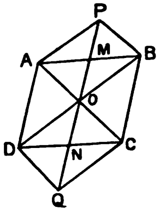

FIG. 140.

9. Since O is the middle pt. of BD, and OF is $\parallel$ to BK,
$\therefore F$ is the middle pt. of DK.
$\therefore \triangle BKF = \triangle FBD$;
$\therefore \triangle FEK$ = sum of $\triangle$s BKF, FEB,
= sum of $\triangle$s FBD, FBC,
$= \triangle DBC$ diminished by $\triangle DFC$,
$= \triangle ABC$ diminished by $\triangle FBC$
(See Ex. 2 on p. 72.)
$= \triangle ABF$.

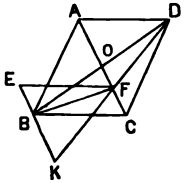

FIG. 141.

10. Let the alternate sides of the polygon *ABCDE* be produced to meet in *F*, *G*, *H*, *K*, *L*.

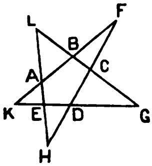

FIG. 142.

Then sum of intr. $\angle$s *ABC*, *BCD*, *CDE*, *DEA*, *EAB*,
= sum of $\angle$ s *BFC*, *FCB*, *CGD*, *GDC*, *DHE*, *HED*, *EKA*, *KAE*, *ALB*, *LBA*,
= sum of $\angle$ s *FCB*, *GDC*, *HED*, *KAE*, *LBA*, together with $\angle$s at *F*, *G*, *H*, *K*, *L*,
= four rt. $\angle$s together with $\angle$s at *F*, *G*, *H*, *K*, *L*.
(I. 32, COR. 2.)

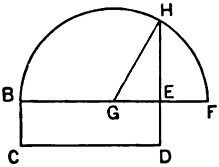

FIG. 143.

11. Take the diagram of II. 14.

Then perimeter of rect. *BEDC* = two *BF*,
= four *BG*,
= four *GH*;
and perimeter of sq. on *HE* = four *HE*;
$\therefore$ since *GH* is greater than *HE*, perimeter of rect. *BCDE* is greater than perimeter of sq. on *HE*.

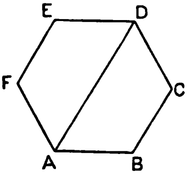

FIG. 144.

12. Let *ABCDEF* be an equiangular hexagon. Join *DA*.

Then $\angle$ *ABC* = $\frac{4}{3}$ of a rt. $\angle$, (P. 55, Ex. 2.)
and $\angle$ *BCD* = $\frac{4}{3}$ of a rt. $\angle$,
and sum of $\angle$s *ABC*, *BCD*, *CDA*, *DAB* = four rt. $\angle$s;
(I. 32, COR. 1.)
$\therefore$ sum of $\angle$s *CDA*, *DAB* = $\frac{4}{3}$ of a rt. $\angle$ = $\angle$ *CDE*;
$\therefore$ $\angle$ *EDA* = $\angle$ *DAB*, and $\therefore$ *ED* is || to *AB*.

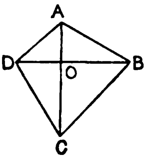

FIG. 145.

13. Let *AC*, *BD*, equal st. lines, intersect at rt. $\angle$s in *O*.

Complete the quadrilateral *ABCD*.

Then twice area of $\triangle$ *ABC* = rect. *AC*, *BO*,
and twice area of $\triangle$ *ADC* = rect. *AC*, *DO*;
$\therefore$ twice area of *ABDC* = rect. *AC*, *BD*,
= sq. on *AC*.

14. (1.) Let $AC=BD$, and $AD=BC$.

Then △s ADC, BDC are equal in all respects.

and ∴ CD is ∥ to AB. (I. 39.)

(2.) Let $AC=BC$, and $AD=BD$.

Produce CD to meet AB in E.

Then, as in (1.), △s ADC, BDC are equal in all respects,

∴ $\angle DCA = \angle BCD$, and ∴ CE is ⊥ to AB.

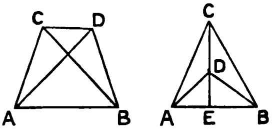

Fig. 146.

15. Bisect $\angle ABC$ by BE, meeting AC in E.

Draw ED ∥ to BC, and ∴ ⊥ to AC.

Then $\angle DEB = \angle EBC$,

$= \angle DBE$;

and ∴ $DE=DB$.

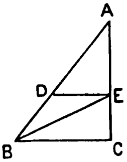

Fig. 147.

16. Let ACB, ADB be any two △s of equal area on the base AB, and on the same side of it.

Join CD. Then, by I. 39, CD must be ∥ to AB.

∴ the locus is a st. line, passing through C, D, ∥ to AB.

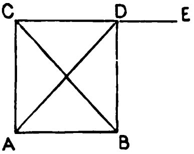

Fig. 148.

17. Let ACB be an isosceles △, and ADB any other △ of equal area on the same base. Join DC and produce it to E; then ECD is ∥ to AB. (I. 39.)

Then ∵ $\angle ECA = \angle CAB$,

and $\angle BCD = \angle CBA$,

∴ $\angle ECA = \angle BCD$;

∴, by Ex. 7, p. 116, the sum of AC, CB is less than the sum of AD, DB;

∴ perimeter of △ ACB is less than perimeter of △ ADB.

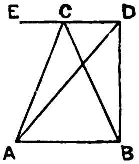

Fig. 149.

18. Let *ABC* be an isosceles $\triangle$, having $\angle$ *BAC* four times as great as either of the other $\angle$s.

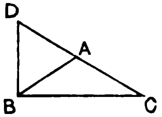

Fig. 150.

Draw *BD* $\perp$ to *BC*, meeting *CA* produced in *D*.
Then $\angle\ ABC = \frac{1}{6}$ of two rt. $\angle$s $= \frac{1}{3}$ of a rt. $\angle$;
$\therefore \angle\ ABD = \frac{2}{3}$ of a rt. $\angle = \frac{1}{3}$ of two rt. $\angle$s.
Also, $\angle\ BAC = \frac{4}{6}$ of two rt. $\angle$s,
and $\therefore \angle BAD = \frac{1}{3}$ of two rt. $\angle$s.
$\therefore \angle\ ADB =$ supplement of sum of $\angle$s *BAD*, *ABD*
$= \frac{1}{3}$ of two rt. $\angle$s,
$\therefore$ *ABD* is an equilateral $\triangle$.

19. Let *BC*, terminated by *AD*, *DE*, two of the sides of $\triangle$ *ADE*, be bisected in *O*. Join *AO*, and produce it to *F*, so that $AO = FO$. Join *BF*, *CF*.

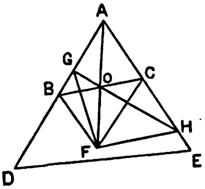

Fig. 151.

Then *ABFC* is a ▱. (P. 59, Ex. 2.)

Again, let *GH* be any other line passing through *O* and terminated by *AD*, *AE*, and, if it be possible, let $GO = HO$. Join *GF*, *HF*.

Then *AGFH* is a ▱. (P. 59, Ex. 2.)

and $\therefore$ *CF*, *HF* are both $\parallel$ to *AB*, which is absurd.

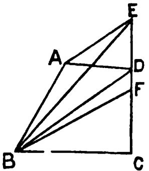

Fig. 152.

20. Let *ABCD* be a quadrilateral. Join *BD*. Draw *AE* $\parallel$ to *BD*, meeting *CD* produced in *E*. Bisect *EC* in *F*. Join *BE*, *BF*. Then *BF* shall bisect the quadrilateral.

For since $\triangle\ BAD = \triangle\ BED$, add to each $\triangle\ BDF$;
then fig. $BADF = \triangle\ EFB$,
$= \triangle\ FBC$, because $FC = FE$.

21. Let *AB*, *CD* be the diagonals. Place them so as to bisect each other at rt. $\angle$s in *O*.

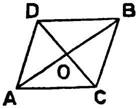

Fig. 153.

Then *ACBD* is the rhombus reqd.
For $\because DO = CO$, and *OB* is common,
and $\angle\ DOB = \angle\ COB$
$\therefore DB = CB$.

Similarly it may be shown that $DB = DA$, and that $CB = CA$.

$\therefore$ *ACBD* is a rhombus.

22. Describe a rectangle $EFGH$ = fig. $ABCD$. (I. 45.)

Draw $AN = \frac{1}{2}$ the given altitude, and $\perp$ to AB.

To AN apply the rect. $ANOP$ = rect. $EFGH$.

Produce OP to Q, making $PQ = 2AN$, and join AQ.

Then $\triangle APQ = \frac{1}{2}$ rect. AP, PQ,
$=$ rect. AP, $AN = ANOP$
$= EFGH = ABCD$.

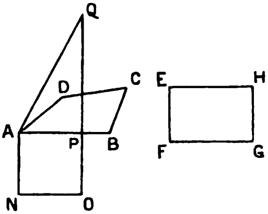

FIG. 154.

23. From O, any pt. in AB, the base of the isosceles $\triangle ACB$, draw OD, OE $\perp$ to AC, BC, and draw AF $\perp$ to BC.

Produce EO to N, meeting AN drawn $\parallel$ to FE.

Then $AFEN$ is a rectangle, and $AF = NE$.

Now $\angle OAN = \angle CBA = \angle OAD$.

Then $\because \angle OAN = \angle OAD$, and
$\angle ODA = \angle ONA$, and AO is common,
$\therefore OD = ON$.

$\therefore AF = NE$ = sum of ON, OE = sum of OD, OE.

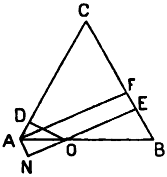

FIG. 155.

24. Sum of sqq. on AC, AD = twice rect. AC, AD with sq. on CD: (II. 7.)

$\therefore$ sqq. on AC, CB, AD = twice rect. AC, AD with sqq. on CD, CB;

$\therefore$ sqq. on AB, AD = twice rect. AC, AD, with sq. on DB. (1.)

Again, sum of sqq. on AB, AE = twice rect. AB, AE with sq. on EB: (II. 7.)

$\therefore$ sqq. on AB, AE, ED = twice rect. AB, AE, with sqq. on EB, ED;

$\therefore$ sqq. on AB, AD = twice rect. AB, AE with sq. on DB. (2.)

$\therefore$, comparing (1) and (2), we have rect. AC, AD = rect. AB, AE.

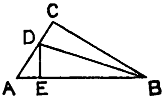

FIG. 156.

25. Let EF bisect the ▱ $ABCD$. Join EB, EC.

Then $\triangle EBC = \frac{1}{2}$ ▱ $ABCD$: (I. 41.)
$\therefore \triangle EBC$ = quadrilateral $EDCF$;
take from each $\triangle EFC$,
then $\triangle EBF = \triangle CED$.

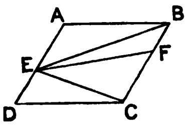

FIG. 157.

26. Draw GM, $FN \perp$ to DB, EC, produced, and draw $AO \perp$ to BC.

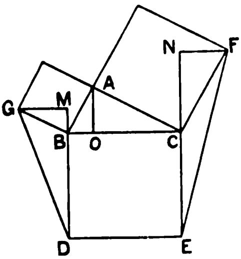

Fig. 158.

Then $\because \angle MBG = \angle ABO$, and
$\angle GMB = \angle AOB$, and $GB = AB$,
$\therefore MB = BO$, and similarly $CN = CO$.
Now sq. on GD
$=$ sqq. on DB, BG with twice rect. DB, BM,
$=$ sqq. on BC, AB with twice rect. BC, BO.
And sq. on EF
$=$ sqq. on EC, CF with twice rect. EC, CN,
$=$ sqq. on BC, AC with twice rect. BC, CO.
$\therefore$ sqq. on GD, EF
$=$ sqq. on BC, AB, BC, AC with twice sq. on BC. (II. 2.)
$=$ five times sq. on BC.

27. Let ABC be any triangle.

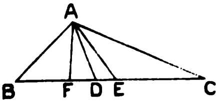

Fig. 159.

Let AD bisect $\angle BAC$, meeting BC in D,
let AF be $\perp$ to BC, meeting BC in F,
let AE bisect BC in E.
Then we can show, as in Ex. 6 on p. 41, that AE is greater than AD.
Also since $\angle AFD$ is a rt. angle, AD is greater than AF.
Again, D must lie between E and F, as is proved in Ex. 3, p. 32.

28. Since AC is bisected in E,
sum of sqq. on AB, $BC =$ twice sum of sqq. on BE, EC: (II. 13, Ex.)
and since AB is bisected in F,

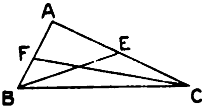

Fig. 160.

sum of sqq. on AC, $CB =$ twice sum of sqq. on CF, FB. (II. 13, Ex.)
$\therefore$ sqq. on AB, AC, BC, $BC =$ twice sqq. on BE, EC, CF, FB;
$\therefore$ three times sq. on $BC =$ twice sqq. on BE, CF with twice sqq. on EC, FB;
$\therefore$ six times sq. on $BC =$ four times sqq. on BE, CF with four times sqq. on EC, FB.
Now four times sqq. on EC, $FB =$ sum of sqq. on AC, AB. (II. 2, Ex.)
$=$ sq on BC;
$\therefore$ five times sq. on $BC =$ four times sqq. on BE, CF.

29. Sq. on AD = sum of sqq. on DE, AE:
sq. on DB = sum of sqq. on DF, BF.
$\therefore$ sum of sqq. on AD, DB = sum of sqq. on DE, AE, DF, BF.

So also, sum of sqq. on AC, CB = sum of sqq. on CE, AE, CF, BF.

Now sum of sqq. on AD, DB = sq. on AB = sum of sqq. on AC, CB;
$\therefore$ sum of sqq. on DE, AE, DF, BF
= sum of sqq. on CE, AE, CF, BF.
$\therefore$ sum of sqq. on DE, DF = sum of sqq. on CE, CF.

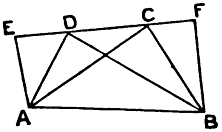

FIG. 161.

30. Since E and G are the middle pts. of AD, AB,
$\therefore GE$ is $\parallel$ to BD. (I. 40, Ex. 1.)
Since F and H are the middle pts. of DC, CB,
$\therefore FH$ is $\parallel$ to BD. (I. 40, Ex. 1.)
$\therefore GE$ is $\parallel$ to FH,
and GH is $\parallel$ to EF. (I. 40, Ex. 1.)
$\therefore GEFH$ is a ▱, and $\therefore GE = HF$.

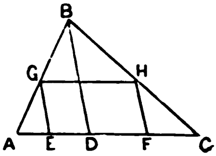

FIG. 162.

31. Draw $AE \perp$ to BC, bisecting BC in E.
Then sq. on AB = sum of sqq. on AE, EB,
and sq. on AD = sum of sqq. on AE, ED;
$\therefore$ difference of sqq. on AB, AD
= difference of sqq. on EB, ED,
= rect. BD, DC. (II. 5.)

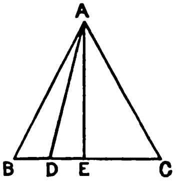

FIG. 163.

32. In AB, BC, CD, DA, the sides of a square, take E, F, G, H equidistant from A, B, C, D. Join $EFGH$.

Then $\because AE = FB$, and $AH = BE$, and $\angle EAH = \angle EBF$,
$\therefore EH = EF$, and $\angle AEH = \angle BFE$.
Similarly, $EH = HG = GF = FE$.
Also, $\angle AEF$ = sum of $\angle$s EBF, BFE,
= sum of $\angle$s EBF, AEH;
and $\therefore \angle HEF = \angle EBF$ = a rt. angle.
$\therefore EFGH$ is a square.

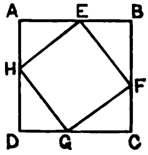

FIG. 164.

33. Let the sides of the equilateral and equiangular pentagon $ABCDE$ be produced to meet in M, N, O, P, Q.

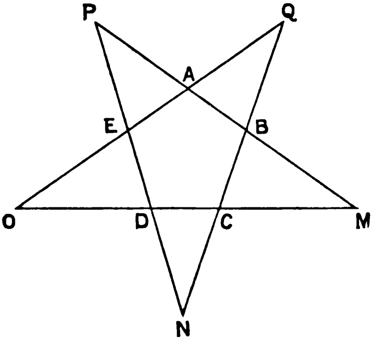

Fig. 165.

Then $\angle\, CBM =$ sum of $\angle$s at P and N,
and $\angle\, BCM =$ sum of $\angle$s at O and Q;
$\therefore$ sum of $\angle$s at P, Q, M, N, $O =$ the three $\angle$s of $\triangle\, BCM$.
$=$ two rt. angles.

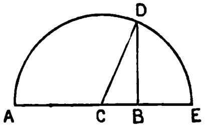

Fig. 166.

34. Let AC, CB, placed in the same st. line, be sides of the unequal squares.
Produce AB to E, making $CE = CA$.
On AE describe a semicircle ADE.
Draw $DB \perp$ to AE. Join CD.
Then sq. on $BD =$ diff. of sqq. on CD, CB,
$=$ diff. of sqq. on AC, CB.

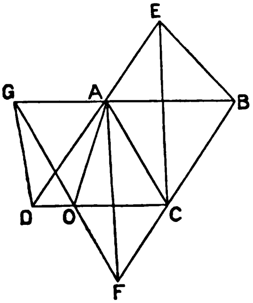

Fig. 167.

35. Join EB, AO.

Then $\triangle\, AEB = \triangle\, EAC$, on same base EA,
$= \triangle\, AFC$, (I. 34.)
$= \triangle\, AOC$, on same base AC,
$= \triangle\, AOG$, (I. 34.)
$= \triangle\, AGD$, on same base AG.

36. Fig. $BADE$ = one-fourth sq. on AD,
= one-eighth sq. on AC,
= one-eighth of sixteen times sq. on AE,
= twice sq. on AE.

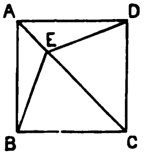

FIG. 168.

37. Bisect $\angle ACB$ by CD, meeting AB in D, and draw $CE \perp$ to AB.

Then $\because \angle DAC = \angle ACD, \therefore AD = CD$;
and $\because \angle BDC$ = sum of $\angle$s DAC, ACD,
$\therefore \angle BDC = \angle DBC$,
and $\therefore CD = BC$, and $\therefore DE = BE$.

Then sq. on AB
= sq. on AC,
= sum of sqq. on AE, EC,
= sq. on AE with difference of sqq. on BC, BE,
= sq. on BC with difference of sqq. on AE, BE,
= sq. on BC with rect. AB, AD,
= sq. on BC with rect. AB, BC.

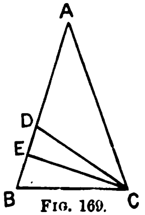

FIG. 169.

38. Let $ABCD$ be a trapezium, having $AB \parallel$ to CD.

Bisect AD in E, and join BE, CE.

Through E draw $FEG \parallel$ to BC, meeting BA, or BA produced, in F, and CD, or CD produced, in G.

Then $\because \angle EFA = \angle EGD$,
and $\angle AEF = \angle GED$, and $AE = DE$;
$\therefore \triangle EFA = \triangle EGD$,
and $\therefore \square\, BFGC$ = trapezium $ABCD$.
Hence $\triangle BEC$ = half of $\square\, BFGC$,
= half of trapezium $ABCD$.

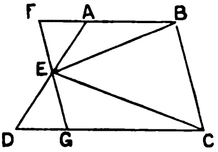

FIG. 170.

39. Let $ABCD$ be the given ▱, and E the line of given altitude.

Draw $CF \perp$ to BC and equal to E.

Through F draw $HFG \parallel$ to AD, meeting BA and CD, or these produced, in H and G.

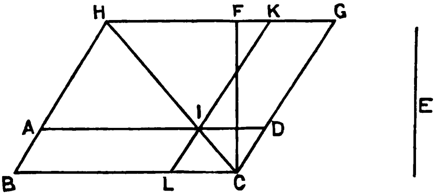

FIG. 171.

Join HC meeting AD in I, and draw $KIL \parallel$ to HB, meeting HG, BC in K and L.

Then $KLCG$ is the ▱ reqd.

For $KD = AL$ (I. 43), and $\therefore KC = AC$.

Also KC, AC are equiangular, and the altitude of KC is $CF = E$.

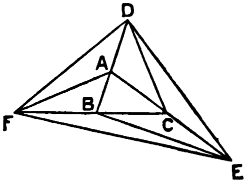

FIG. 172.

40. Let ABC be a △. Produce BA to D, AC to E, and CB to F, each to twice its original length.

Join AF, BE, CD, DF, FE, ED.

Then △ DEF = sum of △s ABC, ABF, ADF, ADC, DCE, CBE, EBF, and each of these △s = △ ABC; (I. 38, Ex. 1.)

$\therefore$ △ DEF = seven times △ ABC.

41. Let ABC be a right-angled △, with ∠ BAC a rt. ∠, and $BA = CA$.

Let BD, the bisector of ∠ ABC, meet AC in D.

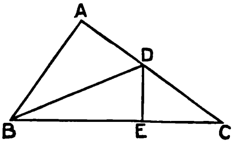

FIG. 173.

Draw $DE \perp$ to BC.

Then $BE = BA$, and $DE = DA$; (I. 26.) and $\because$ ∠ $ECD = \frac{1}{2}$ a rt. ∠, $\therefore$ ∠ $EDC = \frac{1}{2}$ a rt. ∠, and $\therefore ED = EC$.

Then sq. on CD = sum of sqq. on EC, ED,
= twice sq. on ED,
= twice sq. on DA.

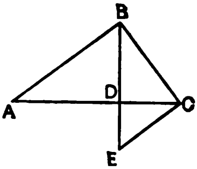

FIG. 174.

42. Rect. *BD*, *DE*=sq. on *CD*; (II. 4, Ex.)
Rect. *AD*, *DC*=sq. on *BD*; (II. 4, Ex.)
∴ rect. *BD*, *DE* with rect. *AD*, *DC*
=sum of sqq. on *CD*, *BD*,
=sq. on *BC*.

43. Let *AB*, *CD* be two st. lines, of which *AB* is the greater.
In *AB* take *AE*=*CD*.
Then sum of sqq. on *AB*, *AE*=twice rect. *AB*, *AE* with sq. on *BE*;
∴ sum of sqq. on *AB*, *CD*=twice rect. *AB*, *CD* with sq. on *BE*;
∴ sum of sqq. on *AB*, *CD* is greater than twice rect. *AB*, *CD*.

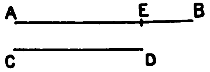

FIG. 175.

44. Let *ABC* be a △, having *AB*=*AC*.
Bisect ∠ *ABC* by *BD*, meeting *AC* in *D*.
Draw *DE* ∥ to *BC*, meeting *AB* in *E*.
Then *EBCD* is the trapezium reqd.
For ∵ ∠ *AED* = ∠ *ABC*, and ∠ *ADE* = ∠ *ACB*;
∴ ∠ *AED* = ∠ *ADE*, and ∴ *AE*=*AD*,
and ∴ *BE*=*CD*.
Again, ∵ *ED* is ∥ to *BC*, ∴ ∠ *EDB* = ∠ *DBC*,
∴ ∠ *EDB* = ∠ *EBD*, and ∴ *ED*=*BE*;
∴ *BE*, *ED*, *DC* are all equal.

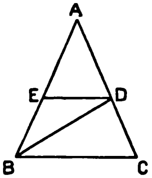

FIG. 176.

45. Since the sum of the squares on the diagonals is equal to the sum of the squares on the four sides (see p. 93, Ex. 2), so long as the sides are of given length the sum of the squares on the diagonals will be the same.

46. Let *AEB* be the equilateral △. Draw *EN* ⊥ to *AB*.
Then *BN* = ½ *AB*=*BC*.
Then area of rectangle = rect. *AB*, *BN*,
and area of triangle = rect. *EN*, *BN*.
Now *EN* is less than *EB*, and ∴ *EN* is less than *AB*;
∴ area of triangle is less than area of rectangle.

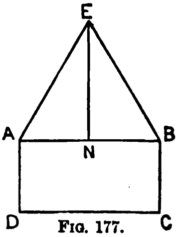

FIG. 177.

47. Draw $BD \perp$ to CO produced.

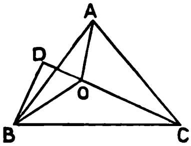

FIG. 178.

Then $\angle BOC = \frac{1}{3}$ of four rt. $\angle$s,
and $\therefore \angle BOD = \frac{1}{3}$ of two rt. $\angle$s;
and $\angle BDO$ is a rt. $\angle$,
and $\therefore BO = 2\,OD$.
(P. 116, Ex. 3.)

Then sq. on BC = sum of sqq. on OB, OC with twice rect. OD, OC; (II. 12.)
$\therefore$ sq. on BC = sum of sqq. on OB, OC with rect. OB, OC.
Similarly, sq. on CA = sum of sqq. on OC, OA with rect. OC, OA;
and sq. on AB = sum of sqq. on OA, OB with rect. OA, OB;
$\therefore$ sum of sqq. on BC, CA, AB = twice sum of sqq. on OA, OB, OC with sum of the rectangles OB, OC; OC, OA; OA, OB.

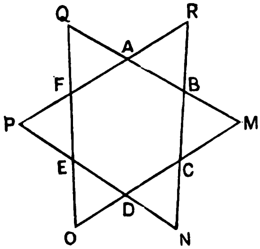

FIG. 179.

48. Let $ABCDEF$ be an equilateral and equiangular hexagon, and let the sides produced meet in M, N, O, P, Q, R.
Then MOQ is a triangle,
and $\therefore$ sum of $\angle$s at M, O, Q = two rt. $\angle$s;
and PNR is a triangle,
and $\therefore$ sum of $\angle$s at P, R, N = two rt. $\angle$s;
$\therefore$ sum of $\angle$s at M, N, O, P, Q, R
= four rt. $\angle$s.

49. Sum of sqq. on BE, ED, DC, with twice rect. BE, ED with twice rect. BE, CD, with twice rect. ED, CD,

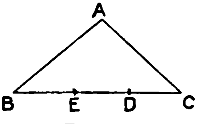

FIG. 180.

= sq. on BC, (See p. 93, Ex. 9.)
= sum of sqq. on BA, AC,
= sum of sqq. on BD, CE,
= sum of sqq. on BE, ED with twice rect. BE, ED, with sum of sqq. on ED, DC with twice rect. ED, CD.

$\therefore$, taking from each common squares and rectangles,
twice rect. BE, CD = sq. on ED.

50. Let $ABCD$ be the given square.

Produce CB to E, making BE = one side of the rectangle.

Complete the rectangle $ABEF$.

Produce FB to meet DC produced in G.

Draw $GHK \parallel$ to CE, meeting AB, FE, produced, in H, K.

Then rect. $BEKH$ = square $ABCD$.

(I. 43.)

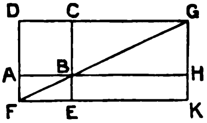

FIG. 181.

51. Sum of sqq. on AD, CD with twice rect. AD, CD,

= sq. on AC,

= sq. on AB,

= sum of sqq. on AD, BD;

$\therefore$ sq. on BD = sq. on CD with twice rect. AD, CD.

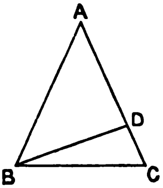

FIG. 182.

END OF BOOK II.

BOOK III.

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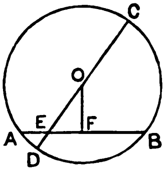

FIG. 183.

EXERCISE 1. Let CD cut AB, but not at right angles.

From O, the centre, draw $OF \perp$ to AB.

Then $AF = BF$, and $\therefore$ AE is not equal to BE.

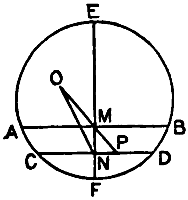

FIG. 184.

Ex. 2. Let EF bisect each of the $\parallel$ chords, AB, CD, in the pts. M, N.

Then the centre of the ⊙ must be in EF.

For, if not, let O be the centre, and join OM, ON, and produce OM to meet CD in P.

Then OM is $\perp$ to AB, and $\therefore$ OP is $\perp$ to CD.

But ON is also $\perp$ to CD, which is absurd.

$\therefore$ centre of ⊙ lies in EF;

$\therefore$ EF bisects AB, CD at rt. $\angle$s.

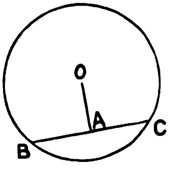

FIG. 185.

Ex. 3. Let A be the given pt.

Find O the centre of the ⊙.

Join OA; draw $AB \perp$ to OA meeting the ○ce in B. Produce BA to meet the ○ce in C.

Then since OA is $\perp$ to BC, it bisects BC.

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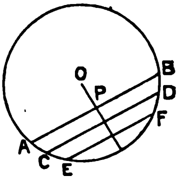

FIG. 186.

EXERCISE 1. From O the centre of the ⊙ draw a st. line OP, bisecting one of the chords, AB.

∴ OP is ⊥ to AB;

∴ OP is ⊥ to the other chords, CD, EF, . . . ;

∴ OP, produced if necessary, bisects all the chords;

∴ the locus is a st. line passing through the centre.

Ex. 2. Let $ABCD$ be a ▱ inscribed in the ⊙ $ABCD$, then must it be a rectangle.

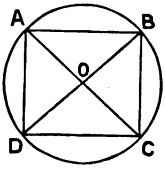

FIG. 187.

For since the diagonals of the ▱ bisect each other in O, O must be the centre of the ⊙, and ∴ $CA=BD$.

Then

∵ $AB=DC$, and BC is common, and $CA=BD$;

∴ $\angle ABC = \angle BCD$;

∴ each of these ∠s is a right angle. (I. 29.)

Similarly, each of the ∠s BAD, ADC is a right angle.

Page 127.

EXERCISE. Let ABC, ABD be two circles cutting one another in the points A and B.

Let O be the centre of ⊙ ABC, and P the centre of ⊙ ABD.

Join OP. Through B draw $CBD \parallel$ to OP.

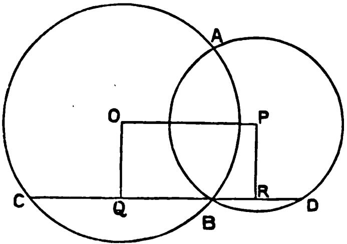

FIG. 188.

Draw OQ, $PR \perp$ to CD.

Then $OPRQ$ is a rectangle,

and ∴ $QR=OP$.

Now $CD=CB+BD$

$=2QB+2BR=2QR=2OP$.

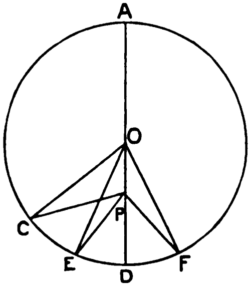

FIG. 189.

Page 130.

EXERCISE 1. Let PE be any other line drawn from P to the ○ce. Join OE.

Then sum of OP, PE is greater than OE;

∴ sum of OP, PE is greater than OD;

∴ sum of OP, PE is greater than sum of OP, PD;

∴ PE is greater than PD.

Ex. 2. Let PE, PC be two positions of PB (fig. to Ex. 1), E being nearer to D than C is.

Join OC, OE.

Then ∵ $CO=EO$, and OP is common, and $\angle COP$ is greater than $\angle EOP$,

∴ CP is greater than EP. (I. 24.)

Ex. 3. Draw OF making $\angle POF = \angle POE$, and join FP (fig. to Ex. 1).

Then ∵ $OE=OF$, and OP is common, and $\angle POE = \angle POF$,

∴ $PE=PF$.

But any other line drawn from P to the ○ce may be shown to be not equal to PE (or PF) as in Ex. 2.

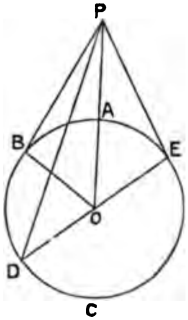

FIG. 190.

Page 131.

EXERCISE 1. Let D be any pt. in the ○ce between B and C.

Then shall PD be greater than PB.

Join BO, DO.

Then ∵ $BO=DO$, and OP is common, and $\angle DOP$ is greater than $\angle BOP$,

∴ PD is greater than PB.

Ex. 2. Make ∠ POE = ∠ POB, and join PE (fig. to Ex. 1).
Then ∵ $BO = EO$, and OP is common, and ∠ BOP = ∠ EOP,

∴ $PB = PE$.

But any other line, drawn from P to meet the ○ce, may be shown not to be equal to PB (or PE) as in Ex. 1.

Page 135.

Exercise. Let ∠ BAC be a right angle.
Make ∠ BAO = ∠ ABC,
and ∴ ∠ CAO = ∠ ACB.
Then $OB = OA = OC$.
∴ O is the centre of the ⊙ described round ABC.

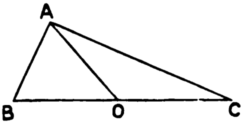

Fig. 191.

Page 137.

Exercise. Let the ⊙, whose centre is A, touch the ⊙s, whose centres are B and C, in D and E respectively.

Then difference between AB and AC,
= difference between DB and EC,
= difference between radii of ⊙s whose centres are B and C,
= half the difference between diameters of those ⊙s.

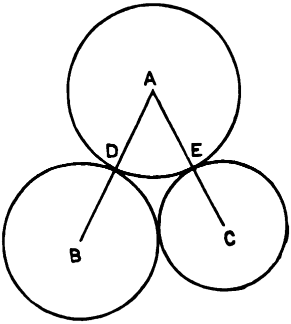

Fig. 192.

Page 140.

Take the diagram of the Proposition, and $OC = 5$ inches and $CQ = 4$ inches.

Then since $\sqrt{25-16} = \sqrt{9} = 3$, ∴ $OQ = 3$ inches;

∴ the second chord is equal to the first.

Page 142.

Exercise 1. Let AB be the given chord, C the given line. Find O the centre of the $\odot$, and draw EOF a diameter.

In EF take $EG = C$, and with centre E and distance EG describe a $\odot$ cutting the given $\odot$ in H. Join EH, and bisect it in M.

With centre O and distance OM describe a $\odot$ cutting AB in N. Through N draw the chord $PR \perp$ to ON.

Then PR and EH being equidistant from the centre O are equal, and $PR = C$, and PR is bisected by AB.

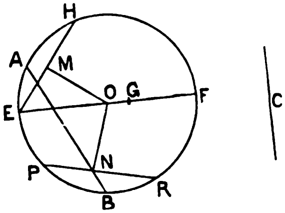

Fig. 193.

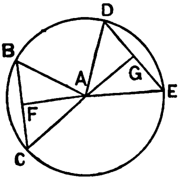

Fig. 194.

Ex. 2. Place the $\triangle$s ABC, ADE so that their vertices coincide in A.

Then since AB, AC, AD, AE are all equal, a $\odot$ described with centre A and distance AB will pass through C, D, E. And since the $\perp$s AF, AG drawn from A to the chords BC, DE, are equal, $\therefore BC = DE$.

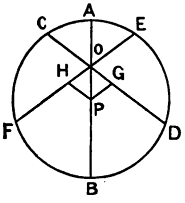

Fig. 195.

Ex. 3. Let AB be a diameter of the $\odot$, and let the chords CD, EF cut AB in O, so that $\angle AOC = \angle AOE$.

From P, the centre, draw PG, PH $\perp$s to CD, EF.

Then $\angle POG = \angle AOC = \angle AOE = \angle POH$.

Then $\because \angle POG = \angle POH$,

and $\angle OHP = \angle OGP$, and OP is common,

$\therefore PG = PH$, and $\therefore CD = EF$.

Page 144.

Exercise 1. Take the diagram of the Proposition, and join OD. Then in $\triangle$s ABO, ADO,

$\because BO = DO$, and OA is common, and $\angle$s OBA, ODA are rt. $\angle$s,

$\therefore AB = AD$.

Ex. 2. Let AB, BC, CD, DA touch the ⊙ in the pts. E, F, G, H.

Then $AE=AH$, $BE=BF$, $CF=CG$, and $DG=DH$.

∴ sum of AB, CD = sum of AE, BE, CG, DG,

= sum of AH, DH, BF, CF,

= sum of AD, BC.

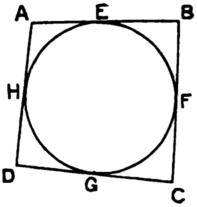

FIG. 196.

Page 145.

EXERCISE. Let A and B be the centres of two ⊙s that touch both the lines CD, EF, which intersect in O.

Then A must lie in the line OA that bisects ∠ DOE, since $OM=ON$, and $AM=AN$.

Similarly B must lie in the line OB bisecting ∠ FOD. And since sum of ∠s EOD, FOD = two rt. ∠s,

∴ sum of ∠s AON, BON, that is, ∠ AOB, is a right angle.

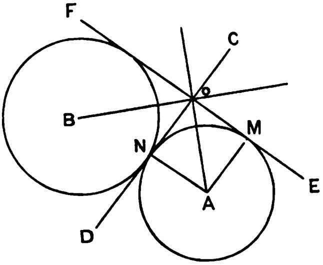

FIG. 197.

Page 146.

EXERCISE. Let ABC, DEF be two concentric ⊙s, and let AC, a chord of the greater, touch DEF in D.

Then shall $DC=DA$.

Find O, the common centre.

Then ∵ $OC=OA$, and OD is common, and rt. ∠ ODC = rt. ∠ ODA,

∴ $DC=DA$.

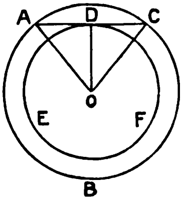

FIG. 198.

Page 148.

Exercise. Let *CB* touch the ⊙ *ABF* in *B*.

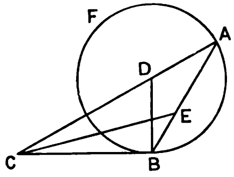

Fig. 199.

Draw *CDA* through *D* the centre. Join *AB*, *DB*.

Bisect *ACB* by *CE*, meeting *AB* in *E*.

Then ∠ *CEB* = sum of ∠s *ACE*, *DAE*,

= sum of $\frac{1}{2}$ ∠ *DCB* and $\frac{1}{2}$ ∠ *BDC*.

Now ∠s *DCB*, *BDC* together = a rt. ∠, because ∠ *DBC* is a rt. ∠ ;

∴ ∠ *CEB* = half a right angle.

Page 151.

Exercise 1. For taking the diagram in the Proposition and joining *OB*, *OC* in fig. 2, the reflex angle *BOC* is double of ∠ *BAC*, and also of ∠ *BDC*, and ∴ ∠ *BAC* = ∠ *BDC*.

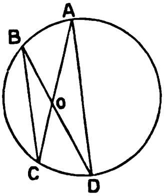

Fig. 200.

Ex. 2. Let *AC*, *BD* be chords intersecting in *O*.

Join *AD*, *BC*.

Then ∵ ∠ *CBO* = ∠ *DAO*, in same segment ;

and ∠ *BCO* = ∠ *ADO*, in same segment :

∴ ∆s *BOC*, *AOD* are equiangular.

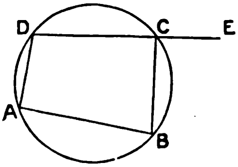

Fig. 201.

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Exercise 1. Let *ABCD* be a quadrilateral inscribed in a ⊙.

Produce *DC* to *E*.

Then ∠ *ECB* = supplement of ∠ *BCD*,

= ∠ *BAD*.

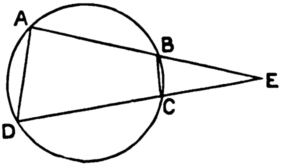

Fig. 202.

Ex. 2. Since

∠ *EBC*=supplement of ∠ *ABC*,
= ∠ *ADE*;
and ∠ *ECB*=supplement of ∠ *BCD*
= ∠ *EAD*,
∴ △s *EBC*, *EAD* are equiangular.

Ex. 3. From the nature of a rhombus, or any ▱ that is not rectangular, two of the opposite ∠s are together greater than two right angles, and therefore it cannot be inscribed in a ⊙. (See also III. 4, Ex. 2.)

Ex 4. Let *ABCD* be a quadrilateral inscribed in a ⊙.

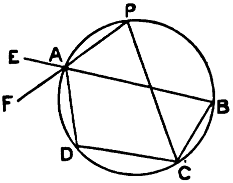

Fig. 203.

Produce *BA* to *E*. Bisect ∠ *BCD* by *CP* meeting the ○ce in *P*. Join *PA* and produce it to *F*.

Then we have to prove that
∠ *EAF*= ∠ *DAF*.

Now ∠ *PCB* = ∠ *PAB* in the same segment,
= ∠ *EAF*;
and ∠ *EAD*=supplement of ∠ *BAD*,
= ∠ *BCD*;
and since ∠ *PCB*=half of ∠ *BCD*,
∴ ∠ *EAF*=half of ∠ *EAD*.

Ex. 5. Join *DE*.

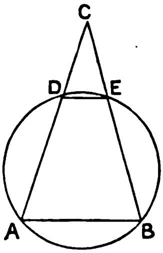

Fig. 204.

Then ∠ *ADE*=supplement of ∠ *ABC*,
=supplement of ∠ *BAC*,
= ∠ *BED*;
∴ ∠ *CDE* = ∠ *CED*, and ∴ *CD*=*CE*.

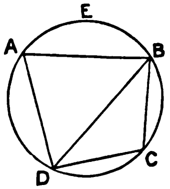

FIG. 205.

Ex. 6. Let *ABCD* be a quadrilateral figure, whose opposite ∠s are together equal to two rt. ∠s. Then a ⊙ *BEDC* described about the △ *BCD* must pass through *A*.

For angle in segment *BED*=supplement of ∠ *BCD*,

= ∠ *BAD*;

∴ *A* must be a pt. in the ○ce.

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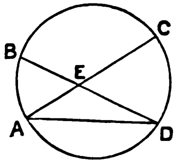

FIG. 206.

EXERCISE 1. Since ∠ *BDA* is of constant magnitude,

and ∠ *CAD* is of constant magnitude,

∴ ∠ *AEB*, which is equal to the sum of ∠s *DBA*, *CAD*, is of constant magnitude.

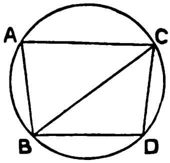

FIG. 207.

Ex. 2. Let *AB*, *CD* be equal arcs.

Join *AB*, *BD*, *DC*, *CA*, *BC*.

Then ∠ *ACB* = ∠ *CBD*, (III. 27.)

and ∴ *AC* is || to *BD*.

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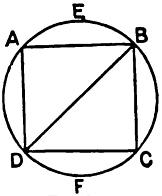

FIG. 208.

EXERCISE 1. Join *BD*.

Then since chord *AB*=chord *DC*,

∴ arc *AEB*=arc *CFD*;

∴ ∠ *ADB* = ∠ *DBC*;

∴ *AD* is || to *BC*.

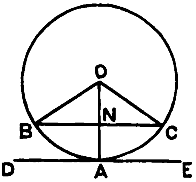

FIG. 209.

Ex. 2. Let DAE touch the ⊙ at A, the middle pt. of arc BAC.

From O, the centre, draw OA; this line is ⊥ to DAE. Let OA cut the chord BC in N.

Then ∵ arc AB=arc AC, ∴ ∠ BON = ∠ CON, and BO=CO, and ON is common to △s BON, CON,

∴ $BN=CN$,

and ∴ OA is ⊥ to BC;

∴ BC is ∥ to DAE.

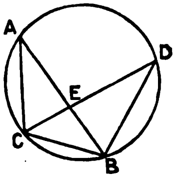

FIG. 210.

Ex. 3. Let AB, CD, equal chords, cut one another in E.

Then ∵ chord AB=chord CD,

∴ arc ACB=arc CBD;

∴ ∠ ACB = ∠ CBD.

Also, ∠ ACD = ∠ ABD, in same segment;

and ∴ ∠ ECB = ∠ EBC;

and ∴ $EB=EC$:

and ∴ $EA=ED$.

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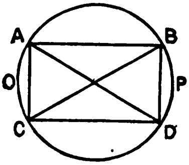

FIG. 211.

EXERCISE 1. Let AB, CD be ∥ chords in the ⊙ $ABCD$.

Join AC, BD, AD, BC.

Then ∵ ∠ ABC = ∠ BCD,

∴ arc AOC=arc BPD;

∴ chord AC=chord BD,

and ∵ ∠ ABD=supplement of ∠ BDC;

= ∠ BAC,

∴ arc ACD=arc BDC;

and ∴ chord AD=chord BC.

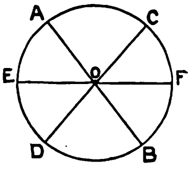

FIG. 212.

Ex. 2. Let AB, CD, EF be three equal chords in the ⊙ $ACBD$, cutting one another in the same pt. O.

Then, by III. 28, Ex. 3,

$OA=OC$, and $OA=OE$;

and ∴ O is the centre. (III. 9.)

Page 160.

EXERCISE. Let *CABD* be a semicircle on the diameter *CD*.

From *O*, any pt. in *CD*, draw *OA* ⊥ to *CD*, and *OB* to the bisection of the ○ce.

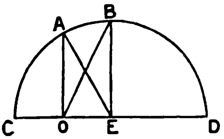

FIG. 213.

Bisect *CD* in *E*, and join *BE*, then *BE* is ⊥ to *CD*. Join *AE*.

Then sq. on *OB*=sum of sqq. on *OE*, *EB*;
and sum of sqq. on *OA*, *OE*=sq. on *AE*;
∴ sum of sqq. on *OB*, *OA*, *OE*=sum of sqq. on *OE*, *EB*, *AE*;
∴ sum of sqq. on *OB*, *OA*=sum of sqq. on *EB*, *AE*,
=twice sq. on radius.

Page 162.

EXERCISE 1. Let *AC* be the diameter of the larger ⊙; and *AB* the diameter of the smaller ⊙.

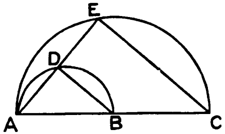

FIG. 214.

Draw *ADE* cutting the smaller ⊙ in *D*, and meeting the larger ⊙ in *E*.

Join *DB*, *EC*.

Then ∠ *ADB*=a rt. ∠ = ∠ *AEC*;
∴ *DB* is ∥ to *EC*.
Now *AC*=twice *AB*;
∴ *AE*=twice *AD*. (See I. 40, Ex. 1.)

Ex. 2. Let *BAC* touch the ⊙ *AED* in *A*.

Let chord *ED* be ∥ to *BAC*.

From *O*, the centre, draw *OA* cutting the chord in *P*.

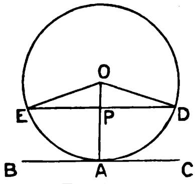

FIG. 215.

Then, since *OA* is ⊥ to *BAC*,
∴ *OP* is ⊥ to *ED*;
∴ *PE*=*PD*;
∴ ∠ *EOP* = ∠ *DOP*;
∴ arc *EA*=arc *DA*.

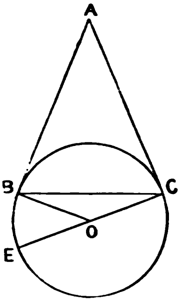

FIG. 216.

Ex. 3. Let AB, AC touch the ⊙ BCE in the pts. B, C. Join BC and through O, the centre, draw the diameter COE.

Now in the quadrilateral $ABOC$, since $\angle$s ABO, ACO are rt. $\angle$s,

∴ $\angle\, BOC$ is the supplement of $\angle\, BAC$.

But $\angle\, BOC$ is the supplement of sum of $\angle$s OBC, OCB;

∴ $\angle\, BAC=$ sum of $\angle$s OBC, OCB;

$=$ twice $\angle\, OCB$.

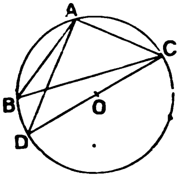

FIG. 217.

Ex. 4. Let ABC be an oblique-angled △, inscribed in a ⊙ ABC.

And let $\angle\, BAC$ be greater than a rt. $\angle$.

Draw COD a diameter: then $\angle\, DAC$ is a rt. $\angle$;

and $\angle\, BAD = \angle\, BCD$, in the same segment,

∴ $\angle\, BAC$ is greater than a rt. $\angle$ by $\angle\, BCD$.

Again, $\angle\, ABC = \angle\, ADC$, in the same segment,

and $\angle\, ADC$ with $\angle\, ACD=$ a rt. $\angle$.

∴ $\angle\, ABC$ is less than a rt. $\angle$ by $\angle\, ACD$.

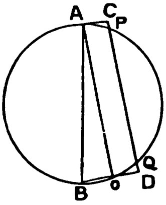

FIG. 218.

Ex. 5. Let AC, BD be the ⊥s on the chord PQ from the extremities of the diameter AOB.

Let BD, the greater perpendicular, cut the ⊙ in O.

Then $\angle\, AOB$, being the $\angle$ in a semicircle, is a rt. $\angle$.

∴ $\angle\, AOD$ is a rt. $\angle$, and ∴, since the $\angle$s at C and D are rt. $\angle$s, $\angle\, CAO$ is a rt. $\angle$.

∴ $ACDO$ is a rectangle, and ∴ $AC=OD$.

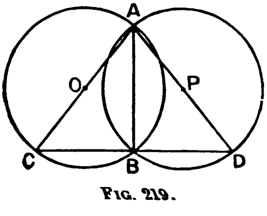

FIG. 219.

Ex. 6. Let the ⊙s ABC, ABD intersect in A and B.

Let O and P be the centres, and AOC, APD diameters.

Then $\angle\, ABC$ is a rt. $\angle$. (III. 31.)

Similarly, $\angle\, ABD$ is a rt. $\angle$;

∴ CBD is a st. line. (I. 14.)

Ex. 7. Let A be the common centre of the ⊙s.

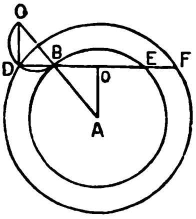

FIG. 220.

Draw AB, a radius of the smaller ⊙, and produce it to C so that $BC=AB$.

On BC describe a semicircle BDC cutting the greater ⊙ in D. Join DB and produce it to meet the original circles again in E and F.

Draw $AO \perp$ to DF, and join CD.

Then $\because$ rt. $\angle\, BDC =$ rt. $\angle\, AOB$,

and $\angle\, ABO = \angle\, CBD$, and $AB=BC$,

$\therefore\ DB=BO$, and $\therefore\ DO=$ twice OB.

Now $FO=DO$, and $EO=OB$:

$\therefore\ FD=$ twice BE.

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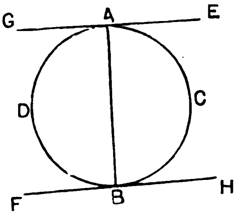

FIG. 221.

EXERCISE. Let GAE, FBH be ∥ tangents to the ⊙ $ACBD$, at the pts. A, B. Join AB.

Then

$\angle$ in segment $ADB = \angle\, EAB$,

$= \angle\, ABF$,

$= \angle$ in segment ACB:

$\therefore$ the angles in these segments are rt. $\angle$s :

$\therefore\ AB$ is a diameter of the ⊙.

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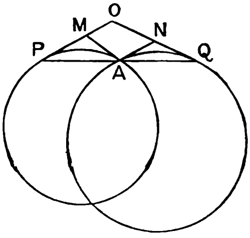

FIG. 222.

EXERCISE 1. Let the tangents at P and Q meet in O, and let the tangents at A meet OP, OQ, in M, N.

Then $\angle\, POQ$

$=$ supplement of sum of $\angle$s MPA, NQA ;

$=$ supplement of sum of $\angle$ s MAP, NAQ ;

$= \angle\, MAN$.

Ex. 2. Let A, B be the given pts., and CD the given line.

Join AB and on it describe a segment of a ⊙ capable of containing the given ∠. The points, where this segment cuts, or the point where it touches, CD, will be points, such that if lines be drawn from them to A and B, the angle contained by these lines will be equal to given ∠.

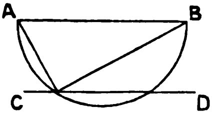

FIG. 223.

Page 165.

EXERCISE 1. Let the ⊙s ABC, ADE touch internally in A.

Draw FAE, cutting smaller ⊙ in C.

Draw the tangent PAQ, touching ⊙s at A.

Then ∠ FAQ = ∠ in segment ABC;

and ∠ FAQ = ∠ in segment ADE:

∴ segments ABC, ADE, contain equal angles.

Similarly segments AGC, AHE, contain equal angles.

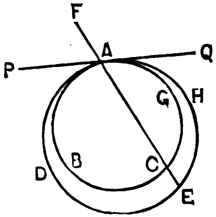

FIG. 224.

Ex. 2. Describe a ⊙ ABC with given radius.

Draw AB cutting off a segment ACB capable of containing an ∠ equal to the given vertical angle, and with centre B and radius = given side, describe a ⊙ cutting ⊙ ABC in D. Join DA, DB.

Then △ DAB is the triangle required.

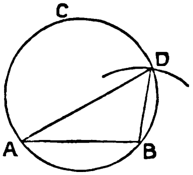

FIG. 225.

Ex. 3. Let AB be the given base. Describe on AB the segment of a ⊙ capable of containing the given vertical angle.

With centre A, and radius = the given ⊥, describe a ⊙, and draw BD touching this ⊙ in D.

Produce BD to meet the segment on AB in C.

Then ACB is the triangle required.

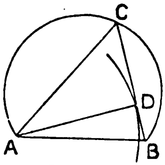

FIG. 226.

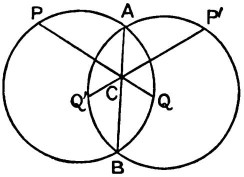

Fig. 227.

Page 166.

Exercise 1. Since rect. CP, CQ = rect. AC, CB;

and rect. CP', CQ' = rect. AC, CB;

$\therefore$ rect. CP, CQ = rect. CP', CQ'.

Ex. 2. Taking diagram of Ex. 1, a $\odot$ described about $\triangle\ PQ'Q$ must pass through P', because rect. CP, CQ = rect. CP', CQ'.

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Miscellaneous Exercises on Book III.

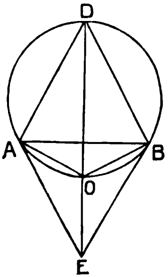


Fig. 228.

1. Let ADB, AOB be the segments. Draw AE, BE tangents to the $\odot$. Join DE, cutting the $\odot$ in O.

Then $\angle\ DOB$ = sum of $\angle$s OBE, OEB,

= sum of $\angle$s BDO, OEB;

and $\angle\ DOA$ = sum of $\angle$s ADO, OEA;

$\therefore \angle\ BOA$ = sum of $\angle$s ADB, AEB

$\therefore$ difference of $\angle$s BOA, ADB = $\angle\ AEB$.

2. Let O, P be the centres of any two of the $\odot$s, A the point of contact. Then the line joining O, P passes through A. (III. 12.) Draw BC a chord of both $\odot$s passing through A. Join OB, CP.

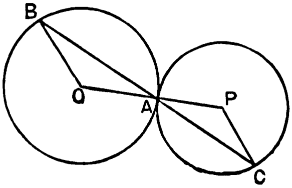


Fig. 229.

Now $\angle\ OBA = \angle\ OAB$ (I. 5.)

$= \angle\ PAC$ (I. 15.)

$= \angle\ PCA$. (I. 5.)

$\therefore OB$ is $\parallel$ to CP.

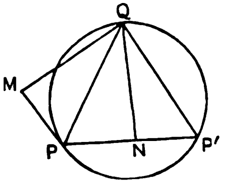


FIG. 230.

3. ∠ *QPM* = ∠ *QP′N* in alternate segment, and ∠ *QMP* = ∠ *QNP′*, each being a rt. ∠ ;
∴ △s *QPM, QP′N* are equiangular.

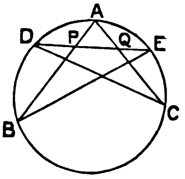


FIG. 231.

4. Let *DE* cut *AB* in *P*, and *AC* in *Q*.
Then ∠ *AQP*=sum of ∠s *ACD, EDC,*
=sum of ∠s *DEB, ABE,*
= ∠ *APQ* ;
∴ *AP*=*AQ*.

5. Let *ABCD* be the quadrilateral fig. inscribed in the ⊙ *ABCD*, and let ∠ *ABC* be a rt. ∠, then ∠ *ADC* is also a rt. ∠. (III. 22.)

From *O*, the centre, draw *OM, ON, OP, OQ* to the middle pts. of the sides.

Then *MQ, NP* are rectangles.

Then sq. on *OB*=sum of sqq. on *OM, OQ,*
and sq. on *OD*=sum of sqq. on *ON, OP* ;
∴ twice sq. on radius=sum of sqq. on the four perpendiculars.

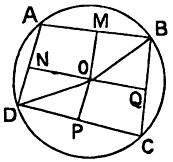


FIG. 232.

6. I. Let *O* and *A* be on the same side of the centre *C*.

Let the chord *BD* be bisected in *O*, and let *EF* be any other chord through *O*.

Draw *CP* ⊥ to *EF*.

Then ∵ ∠ *CPO* is greater than ∠ *COP*,
∴ *CO* is greater than *CP* ;
∴ *FE* is greater than *BD*. (III. 15.)
∴ arc *EAF* is greater than arc *BAD* ;
∴ ∠ *BAD* is greater than ∠ *EAF*.

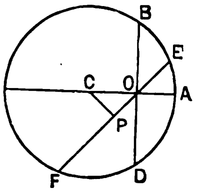


FIG. 233.

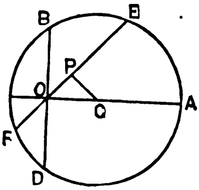


FIG. 233 *a*.

II. Let O and A be on opposite sides of the centre.

Make the same construction.

Then EF is greater than BD :

$\therefore$ arc EBF is greater than arc BFD ;

$\therefore$ arc EAF is less than arc BAD ;

$\therefore \angle BAD$ is less than $\angle EAF$.

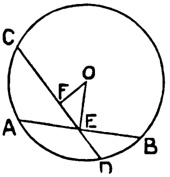


FIG. 234.

7. Let O be the centre of the $\odot$ ABC.

Bisect the chord AB in E, and draw the chord CED.

Bisect CD in F and join OF.

Then $\angle OFE$ is a rt. $\angle$;

$\therefore \angle OFE$ is greater than $\angle OEF$;

$\therefore OE$ is greater than OF,

$\therefore CD$ is nearer the centre than AB is.

Similarly, if a third chord be drawn through F, it may be shown that this chord is nearer to the centre than CD is, and so on.

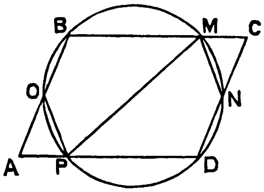


FIG. 235.

8. Let a $\odot$ pass through B, D the opposite $\angle$s of the ▱ $ABCD$. Draw MN, OP joining the pts. of intersection of the $\odot$ and the sides of the ▱. Join MP. Then

$\angle MPO$ = supplement of $\angle MBO$ (III. 22.)
= supplement of $\angle NDP$ (I. 34.)
= $\angle PMN$ (III. 22.)

$\therefore MN$ is $\parallel$ to OP.

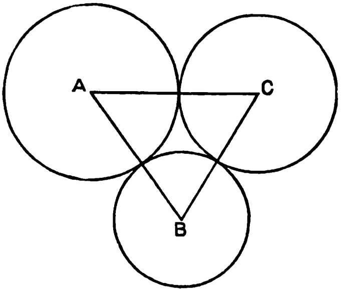


FIG. 236.

9. Let r_1, r_2, r_3 be the radii of the $\odot$s, of which the centres are A, B, C.

Then AB = sum of r_1 and r_2 ;

and AC = sum of r_1 and r_3 ;

$\therefore$ difference of AB, AC = difference of r_2 and r_3, which is independent of r_1, and is therefore invariable, when the circles whose centres are B and C are given.

10. Take any ⊙ *ACBD*, and let *O* be its centre.

Draw the diameters *AOB*, *COD* at rt. ∠s.

Through *A*, *B*, *C*, *D* draw lines ∥ to *CD*, *AB*, meeting in *M*, *N*, *P*, *Q*.

Then *QM*, *MN*, *NP*, *PQ* are tangents to the ⊙ *ACBD*, and are all equal.

Then with centre *O*, and distance *OP* describe a ⊙, which will pass through *P*, *Q* *M*, *N*, and will be the outer circle reqd.

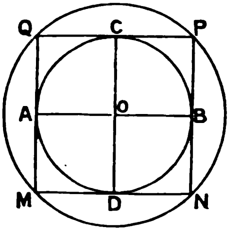


FIG. 237.

11. Let *ABCD* be any quadrilateral inscribed in a ⊙.

Bisect any two adjacent sides, *AD*, *AB* in *E*, *F*.

Let *O* be the centre of the ⊙.

Then *OE*, *OF* are ⊥s on *AD*, *AB*.

∴ since ∠s *AEO*, *AFO* are rt. ∠s, a ⊙ described about △ *AFE* will pass through *O*, and *AO* will be its diameter.

∴ the radius of the ⊙ about *AFE* will be half *AO*; and since *AO* passes through the centre of this ⊙, this ⊙ will touch the ⊙ *ABCD*.

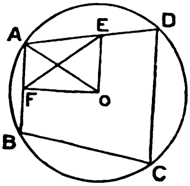


FIG. 238.

Similarly it may be shown that if any other adjacent sides be bisected, and the pts. of bisection joined, the ⊙ described about the triangle thus formed will have a radius half *AO*, and will touch ⊙ *ABCD*.

12. Let *AB* be the given st. line, *MDN* the given ⊙.

From *O*, the centre, draw *OC* ⊥ to *AB*, cutting the ⊙ in *D*.

Through *D* draw *EDF* ∥ to *AB*.

Then ∵ ∠ *EDO* = ∠ *ACO* = a rt. ∠;

∴ *EDF* is a tangent to the ⊙.

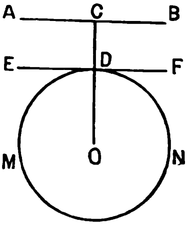


FIG. 239.

13. Let A be the given radius, DE the line in which the centre of the ⊙ is to be, and BC the line which is to be a tangent to the ⊙.

Draw $BF \perp$ to BC, and equal to A.

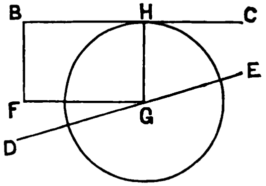


FIG. 240.

Draw $FG \parallel$ to BC, meeting DE in G.

Draw $GH \perp$ to BC.

Then a ⊙ described with centre G and distance GH will touch BC, and will have its radius equal to A.

14. Let AB be the diameter of the given ⊙, and O the centre.

Let CD be the given line. Draw $DE = OB$, and $\perp$ to CD.

Join CE. Produce AB to F, so that $OF = CE$.

Draw FP a tangent to the ⊙ at P.

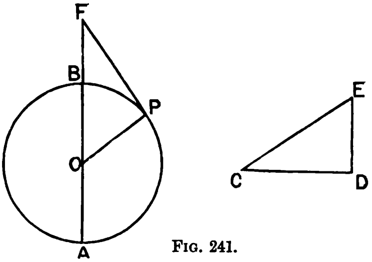


FIG. 241.

Then sum of sqq. on FP, OP = sq. on FO,

= sq. on CE,

= sum of sqq. on CD, ED;

∴ sq. on FP = sq. on CD;

∴ FP = the given line CD.

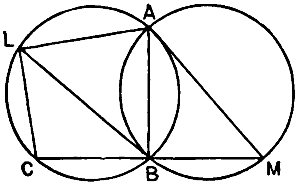


FIG. 242.

15. $\angle ABC$ = sum of $\angle$s BAM, AMB.

Now ∵ $LC = BM$,

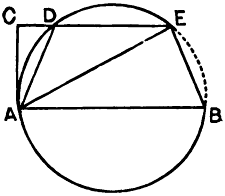
∴ $\angle LBC = \angle BAM$; (III. 28, 27.)

∴ $\angle ABL = \angle AMB$;

∴ arc AL = arc AB. (III. 26).

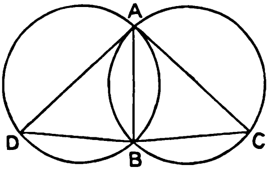
Also, since $\angle LBC = \angle BAM$,

∴ LB is a tangent at B.



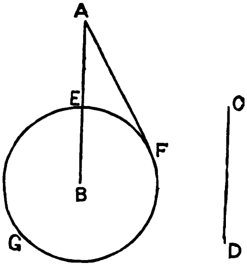
16. Since $\angle DAC = \angle DEA$, (III. 32.)
$= \angle EAB$; (I. 29.)
and $\angle CDA = \angle EBA$; (III. 32.)
$\therefore \triangle$s ACD, EAB are equiangular.

FIG. 243.



17. Since $BC = AB$,
$\therefore \angle CAB = \angle ACB$,
$= \angle ADB$, (III. 28, 27.)
$= \angle$ in segment ADB;
$\therefore$ AC touches the $\odot ABD$.

FIG. 244.



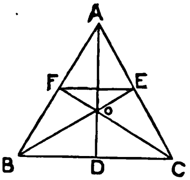
18. Let CD be the given line. Join AB. In AB take a pt. E, such that rectangle BA, AE=sq. on CD. (See p. 120, Ex. 50.)

With centre B and distance BE describe a $\odot$ EFG, and draw AF a tangent to this $\odot$ at F.

Then $\because$ rect. BA, AE = sq. on AF; (III. 37).
$\therefore AF = CD$.

FIG. 245.

19. Let ABC be a $\triangle$. Draw BE, CF $\perp$ to AC, AB, and let them intersect in O. Join AO and produce it to meet BC in D; then shall AD be $\perp$ to BC.



Join EF. Then a $\odot$ may be described about $AEOF$; (III. 22, Ex. 6.)
$\therefore \angle FAO = \angle FEO$. (III. 21.)
Also, a $\odot$ may be described about $BFEC$; (III. 31.)
$\therefore \angle FCB = \angle FEO$.
Hence $\angle FAO = \angle OCD$, and $\angle AOF = \angle COD$,
$\therefore \angle ODC = \angle OFA$ = a rt. $\angle$.

FIG. 246.

20. Let O be the intersection of the $\perp$ s as in Ex. 19.
Then, by II. 7,

sum of sqq. on AD, AO = twice rect. AD, AO with sq. on DO,
sum of sqq. on BE, BO = twice rect. BE, BO with sq. on EO,
sum of sqq. on CF, CO = twice rect. CF, CO with sq. on FO;

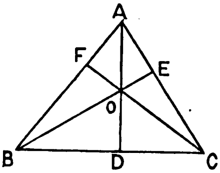


FIG. 247.

∴ sum of sqq. on AD, BO; BE, CO; CF, AO = twice sum of rects. AD, AO; BE, BO; CF, CO; with sum of sqq. on DO, EO, FO;

∴ sum of sqq. on AD, BD, DO; BE, CE, EO; CF, AF, FO = twice sum of rects. AD, AO; BE, BO; CF, CO; with sum of sqq. on DO, EO, FO;

∴ sum of sqq. on AD, BD; BE, CE; CF, AF = twice sum of rects. AD, AO; BE, BO; CF, CO;

∴ sum of sqq. on AB, BC, CA = twice sum of rects. AD, AO; BE, BO; CF, CO.

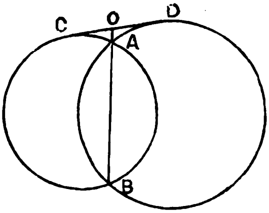


FIG. 248.

21. Let ⊙s ABC, ABD intersect in A and B.

Draw CD a common tangent to the ⊙s in C, D.

Let AB meet CD in O.

Then sq. on OC = rect. BO, OA, (III. 36.)
= sq. on OD;
∴ OC = OD.

22. $\angle CAG = \angle CBD$, in same segment;

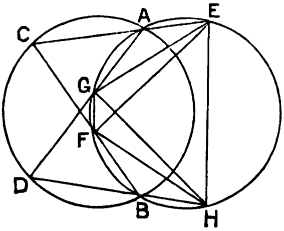
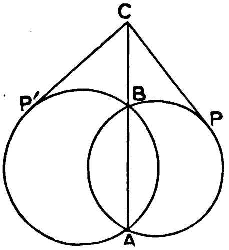


FIG. 249.

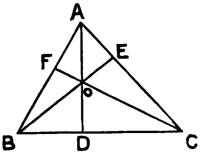
∴ $\angle GAE = \angle FBH$;
∴ $\angle GFE$ = supplement of $\angle GAE$, (III. 22.)
= supplement of $\angle FBH$,
= $\angle HGF$;
and $\angle GEF = \angle GHF$ in same segment;
∴ $\angle EGF = \angle GFH$;
∴ sum of $\angle$s GFH, FHE = two rt. $\angle$s; (III. 22.)
∴ GF is || to EH.



23. Since rect. AC, CB = sq. on CP, (III. 37.)
and rect. AC, CB = sq. on CP',
∴ sq. on CP = sq. on CP';
∴ $CP = CP'$.

FIG. 250.

24. Let ABC be a △, AD, BE, CF the ⊥s from the angular pts. on the opposite sides, intersecting in O.



Then ∵ a ⊙ described on BC as diameter will pass through F and E, (III. 31.)
∴ rect. CO, OF = rect. BO, OE (III. 36.)
Again, ∵ a ⊙ described on AB as diameter will pass through E and D,
∴ rect. BO, OE = rect. AO, OD (III. 36.)

FIG. 251.

25. Let AB, AC be equal st. lines drawn from A to the ⊙ BCD, and let them be produced to meet the ○ce again in D, E.

Draw OM, ON ⊥ to BD, CE, and join OA, OB, OC.

Then ∠ ABO = ∠ ACO; (I. 8.)
∴ ∠ OBM = ∠ OCN, and ∠ OMB = ∠ ONC,
and $OB = OC$;
∴ $OM = ON$, and ∴ AD, AE are equidistant from the centre.

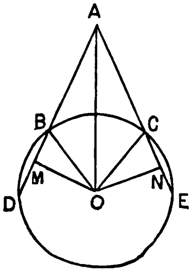


FIG. 252.

26. $\angle CBC' = \angle CBB' + \angle B'BC' = \angle CB'B + \angle BB'C' = \angle CB'C'$,
and $\angle BCB' =$ two $\angle BAB'$,
and four right $\angle$s $- \angle BC'B' =$ two $\angle BA'B'$;

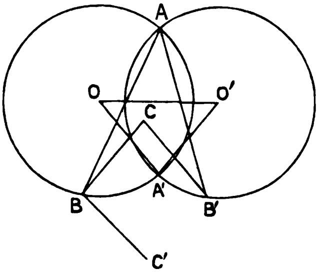


FIG. 253.

$\therefore \angle BCB' + \angle BC'B' =$ four rt. $\angle$s + two $\angle BAB'$ − two $\angle BA'B'$,
= four rt. $\angle$s − two $\angle ABA'$ − two $\angle AB'A'$,
= four rt. $\angle$s $- \angle AOA' - \angle AO'A'$,
= two $\angle OA'O'$;
$\therefore$ two $\angle CBC' =$ four rt. $\angle$s − two $\angle OA'O'$;
$\therefore \angle CBC' =$ two rt. $\angle$s $- OA'O'$.

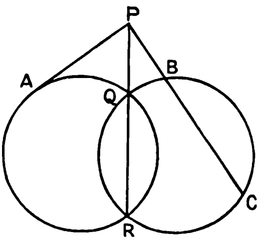


FIG. 254.

27. Let RQ the common chord of the ⊙s AQR, BQR be produced to P. Draw PBC a chord of ⊙ BQR.

Then rect. CP, PB = rect. RP, PQ,
= sq. on PA:

$\therefore$ a ⊙ passing through A, B, C has PA for a tangent;

$\therefore$, PA being the common tangent to this ⊙ and to ⊙ AQR, these ⊙s touch at A.

28. Let AB be the given base, and bisect it in C.

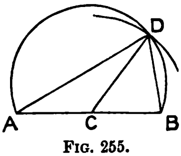
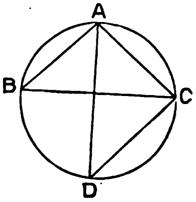


FIG. 255.

On AB describe a segment of a ⊙ ADB capable of containing an angle = given angle.

With centre C and radius = given length of the line from the vertex to the middle pt. of the base describe a ⊙ cutting ADB in D.

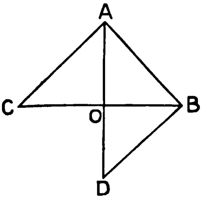
Join AD, DB. Then ADB is the △ reqd.



29. $\angle DCB = \angle BAD$, in same segment,

$= \text{half} \angle BAC$.

FIG. 256.



30. Let O be the intersection of AD, BC.
Then $\because \angle CBD = \angle CAD$,
and $\angle BOD = \angle AOC$,
$\therefore \angle ADB = \angle ACB$; (I. 32.)
$\therefore$ a ⊙ passing through A, B, C will pass through D. (III. 21.)

FIG. 257.

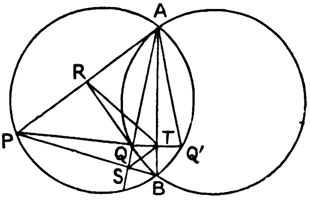


FIG. 258.

31. Obviously $AQ = AQ'$:
$\therefore \angle APQ' = \angle ABQ$; (III. 28, 27.)
$\therefore$ in $\triangle$s APT, ABR,
$\angle ARB = \angle ATP$,
$=$ a right angle.
Hence a ⊙ may be described round $ATQR$;
and $\therefore \angle RTQ = \angle PAQ =$ complement of $\angle APB$ (since $\angle ASP$ is a right angle, for reasons similar to those by which we proved $\angle ARB$ to be a right angle).
Similarly $\angle STQ =$ complement of $\angle APB$:
$\therefore \angle RTQ = \angle STQ$.

32. Let $ABCD$ be a quadrilateral such that, when DA is produced to E, $\angle EAB = \angle BCD$.
Then $\angle BAD$ is the supplement of $\angle BCD$.
$\therefore$ a ⊙ may be described about $ABCD$.
$\therefore \angle BDA = \angle BCA$ in same segment.
Similarly it may be shown that the angles subtended by the other sides are equal.

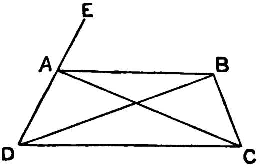


FIG. 259.

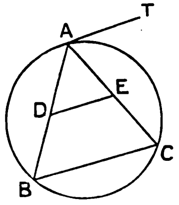


FIG. 260.

33. Describe a ⊙ about the △ ABC.

Draw AT a tangent to this ⊙.

Then $\angle TAC = \angle ABC$, (III. 32.)

$= \angle ADE$: (I. 29.)

∵ AT is a tangent to the ⊙ described about ADE.

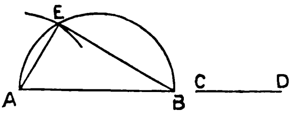


FIG. 261.

34. Let AB and CD be the sides of the given squares, AB being the greater. On AB describe a semicircle, and with centre A and distance CD describe a circle cutting the semicircle in E.

Join AE, BE.

Then sq. on EB = difference of sqq. on AB, AE,

= difference of sqq. on AB, CD.

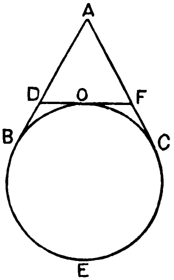


FIG. 262.

35. Let AB, AC be drawn as tangents to the ⊙ BCE from the same pt. A. Draw DOF a tangent to the ⊙ at O, meeting AB, AC in D, F.

Then ∵ $DO = DB$, (III. 17, Ex. 1.)

and $FO = FC$ (III. 17, Ex. 1.)

∴ perimeter of △ ADF = sum of AD, DO, OF, FA,

= sum of AD, DB, FC, FA,

= sum of AB, AC,

which is therefore the same for all positions of O.

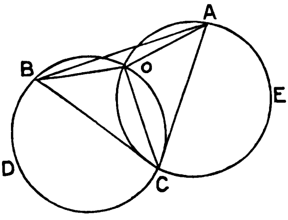


FIG. 263.

36. Let O be the pt. where the circles described on AC, BC intersect.

Join AO, BO, CO.

Then $\angle AOC$ = supplement of $\angle$ in segment AEC,

and $\angle BOC$ = supplement of $\angle$ in segment BDC,

Then $\angle AOB$ will be supplement of $\angle$ in segment of ⊙ described on AB, since sum of $\angle$ s AOB, BOC, AOC = four rt. $\angle$ s.

∴ the ⊙ described on AB will pass through O.

37. Produce *BA* to *D*.
Then

$\angle DAA' = \angle ABC$, (I. 29.)
$=$ supplement of sum of $\angle$ s *BAC*, *ACB*,
$=$ supplement of twice $\angle ACB$,
$=$ supplement of twice $\angle A'AE$, (I. 29.)
Now $\angle AA'E = \angle A'BC$, (I. 29.)
$= \angle A'AE$, (III. 21.)
$\therefore \angle AEA' =$ supplement of twice $\angle A'AE$;
$\therefore \angle DAA' = \angle AEA'$;
and $\therefore$ *BA* is a tangent to ⊙ described about the △ *AEA'*.

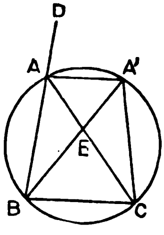


FIG. 264.

38. Let *ABCD* be the given square.
Draw the diagonals intersecting in *O*.
Draw *EOG*, *HOF* || to *AD* and *AB*.
Bisect $\angle$s *EOB*, *BOF*, *FOC*, etc., by *OI*, *OK*, *OL*, etc., as in the diagram.

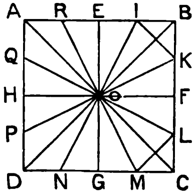


FIG. 265.

Then in △s *OBI*, *OBK*,
$\because \angle IOB = \angle KOB$, and $\angle IBO = \angle KBO$,
and *OB* is common,
$\therefore OI = OK$.
Also in △s *OER*, *OEI*,
$\because \angle EOR = \angle IOE$, and $\angle OER = \angle OEI$,
and *OE* is common,
$\therefore OR = OI$.
Hence *OI*, *OK*, *OL*, etc., are all equal.
Also, since $\angle IOK = \angle KOL$,
$\therefore IK = KL$, and similarly $KL = LM$, etc.
Therefore a ⊙ described with centre *O* and distance *OI* will be the ⊙ reqd., for the arcs subtended by *IK*, *KL*, *LM*, etc., will all be equal.

39. Let *FG* produced meet *AB* in *M*.
Bisect *AB* in *C*, and join *CD*. Bisect *FG* in *O*, and join *OD*, *OE*.
Then $\because \angle$s *ADB* and *AEB* are rt. $\angle$s, (III. 31.)
$\therefore \angle$s *FDG* and *FEG* are rt. $\angle$s;
and $\therefore$ a circle can be described about *DFEG*;
and *FG* is a diameter of this ⊙, and *O* is its centre.
$\therefore OD = OE$.

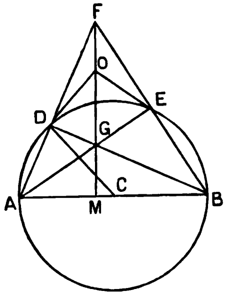


FIG. 266.

Again, since AE is $\perp$ to FB, and BD is $\perp$ to AF, $\therefore$ FM is $\perp$ to AB. (Ex. 19, p. 170.)

Then sum of $\angle$s MFA, DAM = a rt. $\angle$;

and $\angle\,ODF = \angle\,OFD = \angle\,MFA$;

$\therefore$ sum of $\angle$s ODF, DAM = a rt. $\angle$;

$\therefore \angle\,ODF = \angle\,DBA = \angle\,CDB$;

$\therefore \angle\,ODC$ is a rt. $\angle$, and $\therefore$ OD is a tangent to the ⊙ ADB.

Similarly OE is a tangent to the ⊙ ADB.

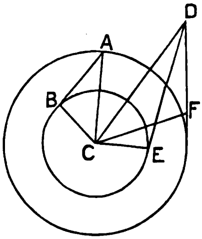


FIG. 267.

40. Let C be the common centre, AB a tangent from any pt. in outer ○ce to the inner circle, DE, DF tangents to the ⊙s from any pt. D.

Join BC, AC, DC, FC, EC.

Then sum of sqq. on FC, DF = sq. on DC,

= sum of sqq. on CE, ED:

$\therefore$ difference of sqq. on DE, DF

= difference of sqq. on FC, CE,

= difference of sqq. on AC, CB,

= sq. on AB.

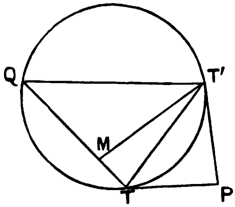


FIG. 268.

41. Sum of $\angle$s $QT'M$, $T'QM$ = a rt. $\angle$.

Now $\angle\,T'QM = \angle\,T'TP$ = half a rt. $\angle$;

$\therefore \angle\,QT'M$ = half a rt. $\angle$;

$= \angle\,T'QM$,

$\therefore T'M = QM$.

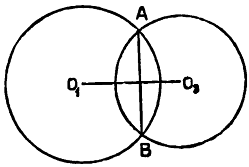


FIG. 269.

42. (1.) Let A and B be the given pts.

Then AB is a common chord of all the circles.

Join O_1, O_2 the centres of any two of the ⊙s.

Then O_1, O_2 bisects AB at rt. angles.

$\therefore$ the locus is a straight line bisecting AB at right angles.

(2.) Draw a diameter AB through the middle pt. of CD any one of the chords, this will pass through the middle pt. of each of the other chords, and will be the locus required.

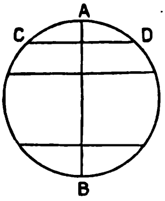


FIG. 270.

(3.) Let AB, AC be the two given lines.

Draw $AP \perp$ to AC and equal to the given st. line. Draw $PDE \parallel$ to AC, meeting BA produced in D.

Then the bisector of $\angle BDE$ is clearly the locus required.

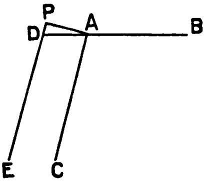


FIG. 271.

(4.) A line drawn $\perp$ to the given line AB, and passing through C the point of contact, will pass through the centre of each circle, and will be the locus required.

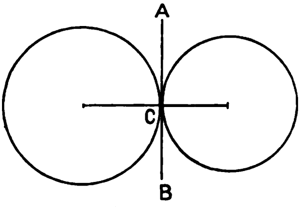


FIG. 272.

(5.) Let O be the pt. through which all the chords pass. From C the centre draw CP, $CQ \perp$ to any two of the chords. Join CO.

Then a $\odot$ described on CO as diameter will pass through P and Q, because the $\angle$s CPO, CQO are rt. $\angle$s.

$\therefore$ this $\odot$ is the locus required.

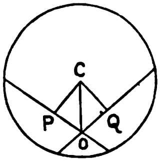


FIG. 273.

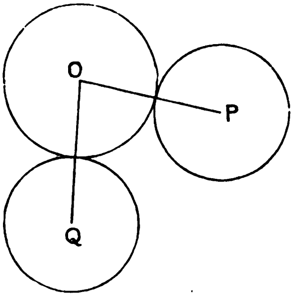


FIG. 274.

(6.) Let O be the centre of the given ⊙.

Then the centres (as P, Q) of all the ⊙s of given radius are equidistant from O, and ∴ the locus is a ⊙ whose centre is O and radius equal to the sum of the radius of the given ⊙ and the radius of any one of the circles of given radius.

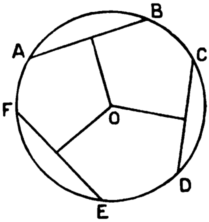


FIG. 275.

(7.) Let AB, CD, EF be any number of equal chords. They are therefore all equidistant from the centre O, that is, ⊥s from O on AB, CD, EF are all equal.

∴ the locus is a ⊙, with centre coinciding with the centre of the original ⊙, and touching all the chords.

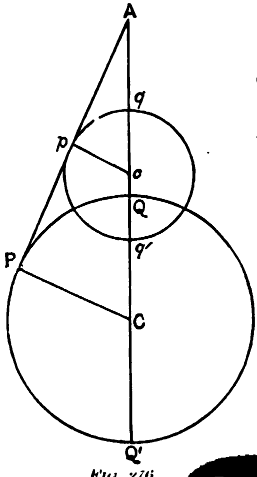


FIG. 276.

(8.) Let A be the given pt., and C the centre of the given ⊙.

Draw $AQCQ'$, make $Aq = \frac{1}{2} AQ$, and $Aq' = \frac{1}{2} AQ'$, and take c the middle pt. of qq'.

Then

$$Ac = \tfrac{1}{2}(Aq + Aq') = \tfrac{1}{4}(AQ + AQ') = \tfrac{1}{2} AC.$$

Similarly, $cq = \frac{1}{2} CQ$.

Draw AP to meet the ⊙ QPQ', and take $Ap = \frac{1}{2} AP$.

Then ∵ $Ap = \frac{1}{2} AP$, and $Ac = \frac{1}{2} AC$,

∴ cp is || to CP, and $cp = \frac{1}{2} CP$.

Hence the locus of p is a ⊙, whose centre is c, and radius one-half of the radius of the given ⊙.

43. On the given base AB describe a segment of a ⊙, ADB, capable of containing an angle equal to the given angle E; this segment will be the locus of the vertex of the △.

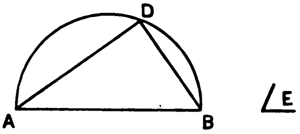


Fig. 277.

44. The locus is an equal circle, whose centre is moved parallel to the given line through a distance equal to the length of the line.

45. Let O be the middle pt. of the ladder AB, DC the vertical wall, EC the horizontal plane.

Then, ∵ $\angle ACB$ is a rt. ∠, a circle described with centre O and distance OA will pass through C.

∴ the locus is the quadrant of a ⊙ described with centre C and distance OA.

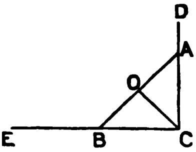


Fig. 278.

46. This is the well-known theorem that the locus is the chord of contact of tangents drawn from A.

Let FH be the chord of contact of tangents from A.

From O, the centre, draw $OP \perp$ to BC, and produce OP to meet HF produced in D.

D shall be the point of intersection of tangents from B and C.

For rect. OM, OA = sq. on OF, since OFA is a rt. ∠.

Also, since ∠s DPA, DMA are rt. ∠s, a ⊙ can be described about A, M, P, D.

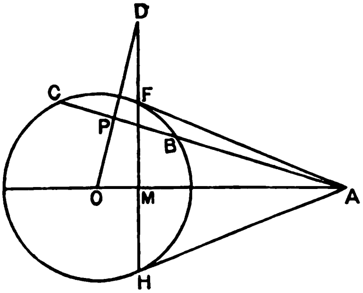


Fig. 279.

∴ rect. OP, OD = rect. OM, OA, (III. 36.)

= sq. on OF,

= sq. on OC;

∴ $\angle OCD$ is a rt. ∠, and CD touches the ⊙ at C.

Similarly BD touches the ⊙ at B.

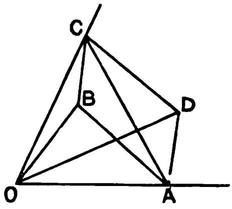


FIG. 280.

47. Since $\angle BCD = \angle COA$,

∴ sum of $\angle$s COA, CDA = two rt. $\angle$s;

∴ a circle may be described round $DCOA$; and this $\odot$ will be of constant magnitude, because AC, a line of constant magnitude, cuts off a segment containing a constant angle.

∴ the constant line CD cuts off a segment containing a constant angle,

∴ $\angle COD$ is a constant angle.

and ∴ D moves along a st. line passing through O.

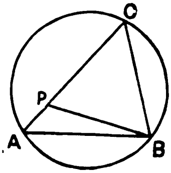


FIG. 281.

48. Since $\angle ACB$ is constant,

∴ $\angle CPB$, which is equal to half the supplement of $\angle ACB$, is constant;

∴ $\angle APB$ is constant;

∴ the locus is the segment of a $\odot$ described on AB, capable of containing an angle = $\angle APB$.

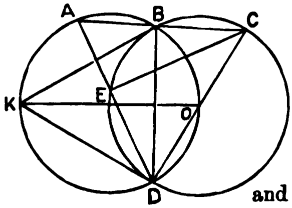


FIG. 282.

49. Let O be the centre of the $\odot$ $CBED$, and KO the diameter of $\odot$ ABD. Then KB, KD are $\perp$ to OB, OD, and ∴ are tangents to the $\odot$ $CBED$.

Now

$\angle ABK = \angle ADK$ in same segment,

and $\angle ABK = \angle BDC$ in alternate segment,

and $\angle ADK = \angle ECD$ in alternate segment,

∴ $\angle BDC = \angle ECD$;

$= \angle EBD$ in the same segment;

∴ BE is $\parallel$ to CD.

Note.—In the diagram CD appears to pass through O, but this is accidental, and is not material to the proof.

50. Let *ABCD* be the ▱, *AB* being the fixed diameter of the ⊙, and *P* the intersection of the diagonals.

The angles at *P* are rt. angles (III. 31), and $AP = PC$, and *BP* is common to △s *APB*, *CPB*;

$\therefore AB = BC$.

∴ *C* lies on a circle whose radius is *AB*, and whose diameter is 2*AB*

Similarly, *D* lies on the ○ce of an equal ⊙.

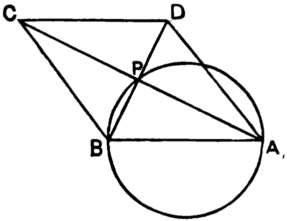


FIG. 283.

51. Let *AB* be the fixed diagonal, and *PQ* the diagonal of constant length.

Let *AP*, *QB* produced meet in *R*.

Then the angles *APB* and *PBQ* are constant.

∴ ∠s *BPR* and *PBQ* are constant;

∴ their difference, ∠ *ARB*, is constant;

∴ *R* lies on the ○ce of a ⊙ passing through *A* and *B*.

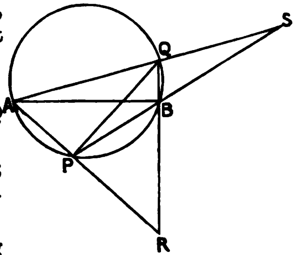


FIG. 284.

Similarly, *S*, the pt. of intersection of *AQ*, *PB*, lies on the ○ce of another circle.

BOOK IV.

Page 180.

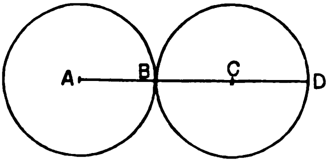


FIG. 285.

EXERCISE. Let A be the given pt. With centre A and radius AB = the given distance, describe a ⊙.

Produce AB to D, so that $BD = 2AB$.

Bisect BD in C, and with centre C and radius CB describe a ⊙. This will be the circle reqd., and BD is ∴ the required diameter.

Page 181.

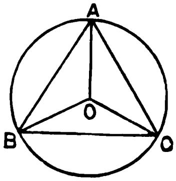


FIG. 286.

EXERCISE. Let ABC be an equilateral △, inscribed in a ⊙, of which the centre is O.

Then ∵ $OA = OB$, and $BC = AC$, and OC is common to the △s AOC, BOC,

$$\therefore \angle OCA = \angle OCB.$$

Similarly for the ∠s at A and B.

Page 183.

EXERCISE 1. For $AD = AF$, AO is common, and $OD = OF$;

$$\therefore \angle DAO = \angle FAO.$$

Ex. 2. Let DEF be a ⊙ inscribed in the right-angled △ ACB, which has ∠ ACB a rt. ∠, and let the ⊙ touch the sides of the △ in D, E, F.

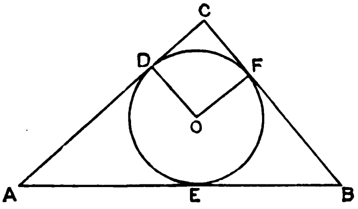


FIG. 287.

Draw the radii OD, OF; these are || to CF, CD, because the ∠s at D, C, F are all rt. angles.

Then

∵ $AD = AE$, and $BE = BF$,

$$\therefore AC + CB - AB = DC + CF,$$
$$= OD + OF,$$
= diameter of ⊙.

Ex. 3. Let *DEF* be a $\odot$ inscribed in the equilateral $\triangle$ *ABC*, touching the sides in *D*, *E*, *F*.

Then $\because \angle OBE = \angle OCE$, (Ax. 7.)

and $\angle OEB = \angle OEC$,

and *OE* is common to $\triangle$s *OBE*, *OCE*,

$\therefore OB = OC$.

Similarly, $OA = OB = OC$.

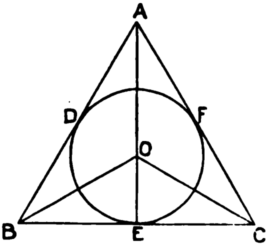


FIG. 288.

Ex. 4. Produce *AB*, *AC*, sides of the $\triangle$ *ABC* to *D*, *E*.

Bisect $\angle$s *DBC*, *ECB* by *BO*, *CO* meeting in *O*.

Draw *OM*, *ON*, *OP* $\perp$s to *BD*, *CE*, *BC*.

Then $\because \angle OBM = \angle OBP$,

and $\angle BMO = \angle BPO$, and *OB* is common,

$\therefore PO = MO$, and similarly $PO = NO$.

$\therefore$ a $\odot$ described with centre *O* and radius *PO* will touch *BD*, *BC*, *CE*.

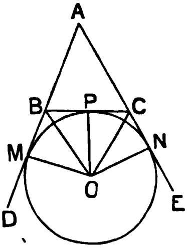


FIG. 289.

Page 185.

EXERCISE. About the $\odot$ *ABCD* describe the square *EFGH*, and draw its diagonals *EG*, *HF*, which intersect in *O* the centre. Then the four $\triangle$s *HOE*, *EOF*, *FOG*, *GOH* are equal in all respects, and it is manifest from Eucl. IV. 4, that if a $\odot$ be inscribed in each of the four $\triangle$s, the $\odot$s so described will be equal, and will touch one another and the $\odot$ *ABCD*.

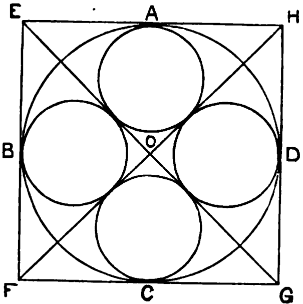


FIG. 290.

Page 186.

EXERCISE 1. From Ex. 2 on p. 144, it is clear that the ▱ must be a square or a rhombus, because in no other ▱ *ABCD* is the sum of *AB* and *CD* equal to the sum of *AD* and *BC*.

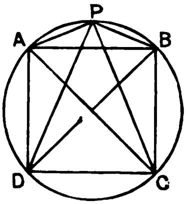


FIG. 291.

Ex. 2. Let $ABCD$ be a square inscribed in a ⊙. From P, any pt. in the Oce, draw PA, PB, PC, PD. Join AC, BD: these are diameters.

Then sum of sqq. on PA, PC = sq. on AC; (I. 47.)

and sum of sqq. on PB, PD = sq. on BD. (I. 47.)

∴ sum of sqq. on PA, PB, PC, PD = twice sq. on diameter.

Page 189.

EXERCISE. Since ∠ BAC = ∠ ACE, for they subtend equal arcs BC, AE,

∴ BA is parallel to CE.

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Miscellaneous Exercises on Book IV.

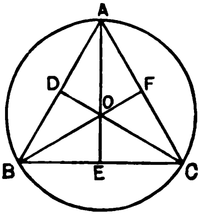

FIG. 292.

1. Let ABC be an equilateral △, and O the centre of the ⊙ described about it. CD, AE, BF the ⊥s from the angular points on the opposite sides pass through O, and bisect the sides.

Then in the right-angled △s EOC, FOC,

∵ $FC = EC$ and OC is common,

∴ $OF = OE$; and similarly $OD = OF = OE$.

2. Let $ABCDEFGH$ be a regular octagon.

Bisect the ∠s HAB, ABC by AO, BO, meeting in O, and join OH.

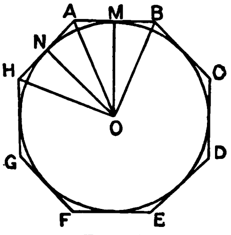

FIG. 293.

Then, as in IV. 13, we can show that ∠ AHO = ∠ ABO;

∴ OH bisects ∠ GHA,

and, if we draw OM, ON ⊥s to AH, AB, we can show that $ON = OM$.

Similarly, if ⊥s be drawn from O to the other sides of the octagon, these will all be equal.

∴ a ⊙, described with centre O and radius OM, will touch each side of the octagon.

3. Since BD touches the ⊙ ACD (see diagram to IV. 10),
$\therefore \angle BDC = \angle CAD$ in alternate segment.
Now $\angle BCD = \angle CBD = 2 \angle CAD$;
$\therefore \angle BCD = \angle CBD = 2 \angle BDC$;
$\therefore BDC$ is the triangle required.

4. Let $ABCD$ be the given rectangle.
The diagonals are equal, and bisect each other in O (I. 34, Ex. 1 and 2), and $\therefore$ a ⊙ described with centre O and radius OA will pass through A, B, C, D.

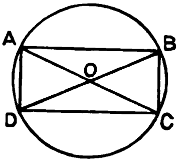

FIG. 294.

5. Describe a ⊙ about the isosceles △ ABC, which has $\angle BAC$ double of each of the angles at the base.
Then $\because \angle BAC =$ sum of $\angle$s ABC, ACB,
$\therefore \angle BAC$ is a right angle;
and $\therefore BC$ is the diameter of the ⊙ described about △ ABC.

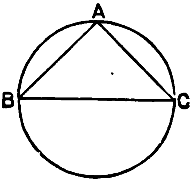

FIG. 295.

6. Let ABC be an equilateral △ described about the ⊙ DEF, touching the ⊙ at the points D, E, F.
Join DE, EF, FD.
Then $\because AD = AE$,
$\therefore \angle ADE = \angle AED$.
Now $\angle DAE = \frac{1}{3}$ of two rt. $\angle$s,
$\therefore \angle ADE$ is $\frac{1}{3}$ of two rt. $\angle$s,
and ADE is an equilateral △.
$\therefore DE$ is || to BC.
Also, AC is bisected in E,
and $\therefore BC = 2 . DE$.

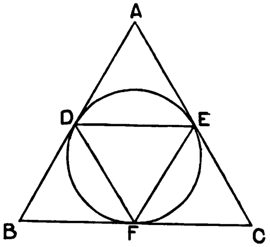

FIG. 296.

7. Let $ABCD$ be a quadrilateral figure, whose diagonals intersect in P, and let rect. AP, PC = rect. BP, PD.

From M, N the middle pts. of AC, BD, draw $OM, ON \perp$ to AC, BD, meeting in O.

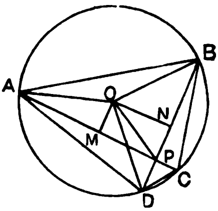

FIG. 297.

Join OA, OB, OP, OD.

Then

rect. AP, PC with sq. on MP = sq. on AM;

and $\therefore$ rect. AP, PC with sqq on MP, OM = sqq. on AM, OM;

$\therefore$ rect. AP, PC with sq. on OP = sq. on OA.

So also,

rect. BP, PD with sq. on OP = sq. on OB.

$\therefore OA = OB$.

Now $OD = OB$ (I. 4), and similarly $OA = OC$,

$\therefore$ a ⊙ described with centre O and radius OA will pass through A, B, C, D.

8. Let ABC be an equilateral △ inscribed in a ⊙, of which O is the centre.

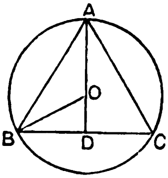

FIG. 298.

Produce AO to D, then AD is $\perp$ to BC.

Then $\angle OBD = \frac{1}{2} \angle ABC = \frac{1}{3}$ of a rt. $\angle$;

$\therefore \angle BOD$ = twice $\angle OBD$;

$\therefore BO$ = twice OD. (Ex. 3, p. 116.)

Now

sq. on AB = sq. on AO + sq. on BO + 2 rect. AO, OD, (II. 12.)

= sq. on AO + sq. on AO + rect. AO, BO,

= three times sq. on AO.

And AO is equal to the side of the hexagon inscribed in the ⊙.

9. Let $ABCD$ be the given rhombus.

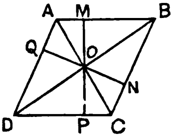

FIG. 299.

Draw the diagonals AC, BD, and from O the point of intersection draw $OM, ON, OP, OQ \perp$ to the sides.

Then $\because \angle OMB = \angle ONB$, and $\angle OBM = \angle OBN$, and OB is common to the △s OMB, ONB;

$\therefore OM = ON$.

Similarly, $OM = OQ = OP$.

$\therefore$ a ⊙ described with centre O and radius OM will touch the sides.

10. Let O be the centre of the ⊙.

Join AO, BO, and let MO cut AB in P.

Then rt. ∠ MAO = rt. ∠ MBO, and $MA = MB$, and $AO = BO$,

$$\therefore \angle AMO = \angle BMO.$$

Hence $BP = AP$, and the ∠s at P are rt. ∠s;

∴ COP is a straight line, because the line drawn from an angular pt. of the equilateral △ through the centre of the ⊙ described about the △ bisects the opposite side at rt. angles.

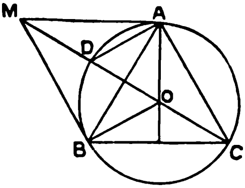

FIG. 300.

Next,

since ∠ MAB = ∠ $ABC = \frac{1}{3}$ of two rt. ∠s,

∴ ∠ $MBA = \frac{1}{3}$ of two rt. ∠s,

and ∴ ∠ $AMB = \frac{1}{3}$ of two rt. ∠s;

∴ ∠ AMD = ∠ ACD; (Ax. 7.)

and ∠ MAD = ∠ ACD in alternate segment,

∴ ∠ AMD = ∠ MAD, and ∴ $AD = MD$.

And ∵ ∠ $AMD = \frac{1}{3}$ of a rt. ∠,

∴ ∠ $MAD = \frac{1}{3}$ of a rt. ∠;

and ∴ ∠ $DAO = \frac{2}{3}$ of a rt. ∠,

= ∠ AOD;

∴ $DA = DO$;

∴ $MD = DO = OC$.

11. Let A, B, C, D, E, F be the angular pts. of the regular hexagon inscribed in the ⊙. Then these are the pts. in which the sides of the circumscribed hexagon touch the ⊙. Take O the centre. Then since the ∠s OAN, OBN are right ∠s, a circle can be described about $ANBO$, of which ON is the diameter.

Bisect ON in P.

Now AOB is an equilateral △, and ON bisects ∠ AOB (I. 8), and is ⊥ to AB.

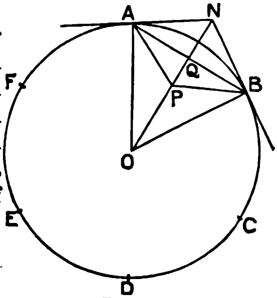

FIG. 301.

Hence $PA = PB$ (I. 4), and $\angle APO = \angle BPO$,
and $\therefore \angle APQ = \angle BPQ$, and each of these $\angle$s$= 2 \angle AOP = \frac{2}{3}$ of a rt. $\angle$;

$$\therefore \angle PAQ = \tfrac{1}{3} \text{ of a rt. } \angle = \angle PAO;$$

$$\therefore \angle QAN = \tfrac{1}{3} \text{ of a rt. } \angle.$$

Hence, since PAN is an equilateral $\triangle$, AQ bisects PN.

Then area of external hexagon $= 12 \,.\, \triangle AON = 6 \,.$ rect. AQ, ON;
and area of internal hexagon $= 12 \,.\, \triangle AOQ = 6 \,.$ rect. AQ, OQ;
and $\therefore$, since $OQ = \frac{3}{4} ON$,
area of internal hexagon $= \frac{3}{4} \,.$ area of external hexagon.

12. Take AB as the diameter of the semicircle, and bisect it in C. Take $DE = 5\ BC$, and on DE describe a rectangle $DEMR =$ sq. on BC.

Take in DE the part $DF = BC$, and draw $FG \parallel$ to DR. Then $DFGR = \frac{1}{5}$ sq. on BC.

Describe a square $S =$ rectangle $DFGR$, and in CB take $CH =$ a side of this square.

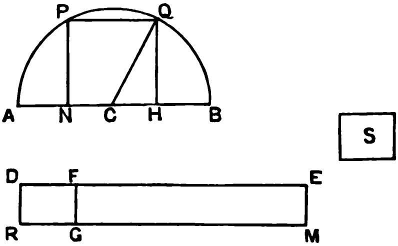

FIG. 302.

Draw $HQ \perp$ to CB.

Then sq. on $CQ = 5$ sq. on CH,
$\therefore$ sq. on $HQ = 4$ sq. on CH, (I. 47.)
$\therefore HQ = 2\ CH$.

Take $CN = CH$, and draw $NP \perp$ to AC.

Then $PN = QH$, and $\therefore QP = NH$.

Then $PNHQ$ is the square required.

13. Let A, B, C, D, E, F be the angular pts. of a regular hexagon inscribed in the given ⊙, whose centre is O.

Produce OD to M, so that $DM=DO$; and produce OE to N, so that $EN=EO$.

Join MN, bisect it in P, and join OP.

Then since $\angle EOD=\frac{1}{3}$ of two rt. ∠s, and $ON=OM$, $\therefore ONM$ is an equilateral △, and $\therefore OP$ is $\perp$ to MN.

∴ circles described with centres M, N, and radius $=DO$, will touch each other at P, and will touch the given ⊙ at D and E.

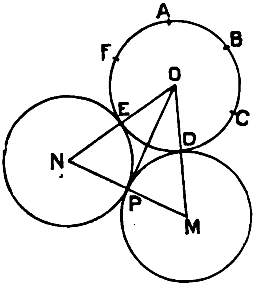

FIG. 303.

Similarly, by producing OF, OA, OB, OC the four other ⊙s may be described.

14. Let OM, ON, OP be the three perpendiculars, so placed that $\angle MON$ is the supplement of one of the given angles, and $\angle PON$ the supplement of another of the angles; then $\angle POM$ must be the supplement of the third angle.

Draw AC, CB, BA $\perp$ to OM, OP, ON, meeting in A, B, C. Then ABC is the triangle required, because the ∠s at A, B, C are supplementary to ∠s MON, NOP, POM respectively.

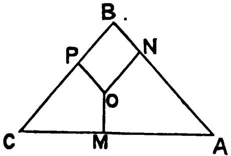

FIG. 304.

15. Let AB, AC be the given lines.

Bisect $\angle BAC$ by AE. Draw $AD \perp$ to AB, and make $AD =$ the given radius.

Draw $DO \parallel$ to AB, and let it meet AE in O, and draw OQ, OR $\perp$ to AB, AC.

Then $\because \angle OQA=\angle ORA$, and $\angle QAO=\angle RAO$, and AO is common;

$$\therefore OR=OQ=AD.$$

∴ a ⊙ described with centre O and radius $=AD$, will touch AB, AC in Q, R.

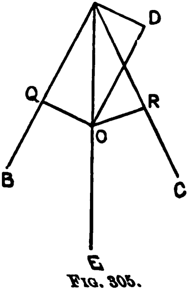

FIG. 305.

16. Let O, C be the centres of the two circles.

Draw OP a tangent to the other ⊙, then since the ⊙s cut one another at rt. ∠s, P is a pt. in the ○ce of the circle whose centre is O.

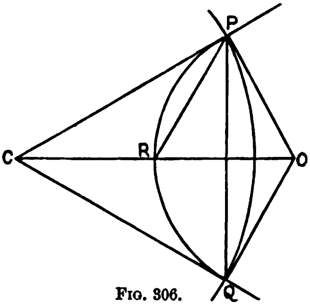

Fig. 306.

Draw OQ, the other tangent, from O to the larger ⊙.

Then since $OC = 2 \,.\, OP$, and OPC is a right ∠, ∴ OC is bisected by the smaller ⊙ in R.

∴ R is the centre of a ⊙ passing through $CPOQ$.

∴ $RP = RO = OP$, and ∴ PRO is an equilateral △.

∴ $\angle POR = \frac{1}{3}$ of two rt. ∠s ;

∴ $\angle POQ = \frac{1}{3}$ of four rt. ∠s ;

and $\angle PCQ = \frac{1}{3}$ of two rt. ∠s ;

∴ PQ is the side of a regular hexagon in the larger circle, and PQ is the side of an equilateral △ in the smaller circle.

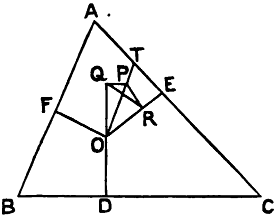

Fig. 307.

17. Let OP meet AC in T.

The angles at Q and R being right ∠s, a ⊙ will circumscribe $PQOR$.

∴ $\angle OQR = \angle OPR = \angle OTE = \angle BAC$.

Also, $\angle QOR$ = supplement of $\angle EOD$, $= \angle ACB$;

∴ the third $\angle ORQ = \angle ABC$;

∴ △s OQR, ABC are equiangular.

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Senate-House Riders.

1848. I. 34. Let $ABCD$ be a ▱, and let the diagonals intersect each other in E.

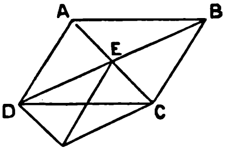

Fig. 308.

Then △ ADC = half the ▱,
= △ DAB.

Take from each △ AED.

Then △ DEC = △ BEA.

Similarly △ AED = △ BEC.

Next, draw DP ‖ to EC and CP ‖ to ED, and join EP.

Then $DP=EC=EA$, and ED is common to $\triangle$s PDE, ADE, and $\angle EDP = \angle AED$;

$$\therefore \triangle DEP = \triangle AED.$$

But $\triangle DEP$=half the ▱ $DECP = \triangle DEC$.

$$\therefore \triangle DEC = \triangle AED = \triangle BEC = \triangle BEA.$$

The diagonal bisects the angles when all the sides are equal.

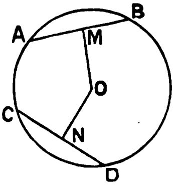

Fig. 309.

III. 15. Let AB, CD be any two equal chords in a circle. Then they are equidistant from O the centre. Draw $\perp$s OM, ON to AB, CD. Then a ⊙ described with centre O and distance OM will touch each of the equal chords.

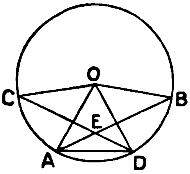

Fig. 310.

III. 20. $\angle AOC =$ twice $\angle ADC$;

$\angle BOD =$ twice $\angle BAD$;

$\therefore$ sum of

$\angle$s AOC, BOD=twice sum of $\angle$s ADC, BAD

=twice $\angle AEC$.

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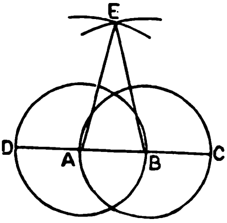


Fig. 311.

1849. I. 1. Let AB be the given line.

With centre A and distance AB describe a ⊙, and produce BA to meet this ⊙ in D.

With centre B and distance BA describe a ⊙, and produce AB to meet this ⊙ in C.

Then with distance A and radius AC describe a ⊙, and with distance B and radius BD describe a ⊙, and from E, where these ⊙s intersect, draw the st. lines EA, EB.

Then EAB is the $\triangle$ required.

II. 11. Taking the diagram on p. 89, it is clear that

(1.) rect. *DC*, *CK*=sq. on *DK*.
(2.) rect. *DF*, *FA*=sq. on *AD* (since *AD*=*AB*).
(3.) rect. *KG*, *GH*=sq. on *HK*.
(4.) *BE* is similarly divided, but the proof depends on Book VI., since if *M* be the pt. where *BE* cuts *HK*, △s *BHM*, *BAE* are similar, and ∴ *AB* and *BE* are similarly divided in *H* and *M*.

IV. 4. For this problem of the escribed circle see IV. 4, Ex. 4.

1850. I. 34. See Exercises 6, 5, 2 on p. 59.

II. 14. See Exercise 50 on p. 120.

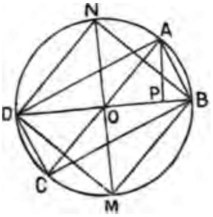


FIG. 312.

III. 31. Let *ABCD* be any rectangle inscribed in a ⊙; then the diagonals bisect each other in the centre *O*.

Draw the diameter *NOM* ⊥ to *BD*; then *NBMD* is a square, and its area is=rect. *NO*, *BD*. Draw *AP* ⊥ to *BD*.

Now area of rectangle *ABCD*=*AP*, *BD*; and since *AP* is less than *AO*, *AP* is less than *NO*, and ∴ area of the square is greater than area of the rectangle.

III. 34. Construct an equilateral △ *ABC*, and draw *AD* ⊥ to *BC*.
Then ∠ *BAD*=$\frac{1}{6}$ of two rt. ∠s.
Draw *HEG* a tangent to the ⊙ *MEF*.
Make ∠ *FEG* = ∠ *BAD*.

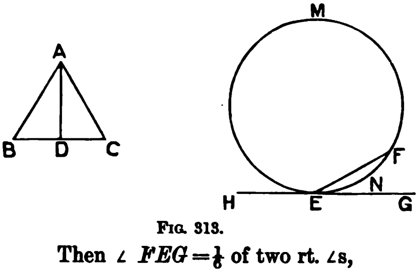


FIG. 313.

Then ∠ *FEG*=$\frac{1}{6}$ of two rt. ∠s,
and ∠ *FEH*=$\frac{5}{6}$ of two rt. ∠s;
∴ ∠ in segment *FNE*=five times ∠ in segment *FME*.

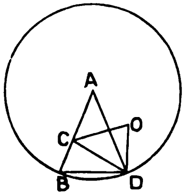


FIG. 314.

IV. 10. If O be the centre of the smaller ⊙,
$\angle COD = 2 \angle CAD$.
Now $\angle CAD = \frac{1}{5}$ of two rt. ∠s;
$\therefore \angle COD = \frac{1}{5}$ of four rt. ∠s.
Hence CD, to which BD is equal, is the side of a regular pentagon inscribed in the smaller ⊙.

1851. I. 38. Let ABC, ABD be two equal △s upon the same base AB, and on opposite sides of it: join CD meeting AB in E.

Make $\angle ABN = \angle ABD$, and $BN = BD$, and join CN, AN, EN.

Then $\triangle ABN = \triangle ABD = \triangle ABC$.

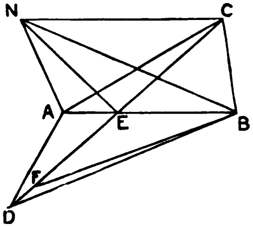


FIG. 315.

Hence, by applying the Proposition as in I. 39 and 40, we obtain that CN is ∥ to BA.

$\therefore \triangle CEB = \triangle NEB$, on same base,
$= \triangle DEB$, by the construction of the figure.

Hence △s CEB, DEB must be on equal bases, CE, DE.

For, if not, let $CE = EF$.

Then $\triangle CEB = \triangle BEF$ (I. 38), which is impossible;
$\therefore CE = DE$.

I. 47. See Exercise 28 on p. 118.

III. 22. From any angular pt. A_1 draw $A_1 A_4$, $A_1 A_8$ to the *third* angular pt. on each side of A_1, cutting off quadrilaterals. Since the number of remaining sides between A_4, A_8 is even, a certain number of quadrilaterals may be formed by joining every second angular pt. from A_4 with A_1, and all these quadrilaterals are inscribed in the ⊙.

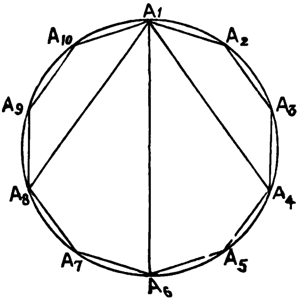


FIG. 316.

$\therefore \angle A_1A_2A_3 + \angle A_3A_4A_1 =$ two rt. angles,
$\angle A_1A_4A_5 + \angle A_5A_6A_1 =$ two rt. angles,
$\angle A_1A_6A_7 + \angle A_7A_8A_1 =$ two rt. angles,
$\angle A_1A_8A_9 + \angle A_9A_{10}A_1 =$ two rt. angles.

Adding, sum of alternate angles A_2, A_4, A_6, etc. = twice as many rt. angles as there are quadrilaterals = 2 $(r-1)$ rt. angles, the number of sides of the polygon being $2r$.

∴ sum of alt. angles + two rt. ∠s = as many rt. ∠s as there are sides.

IV. 16. Describe a regular quindecagon in a ⊙.

Each of its angles = $\frac{13}{15}$ of two rt. ∠s. (I. 32, Cor. 1.)

Produce one of the sides; then the angle between the produced part and the adjacent sides = $\frac{2}{15}$ of two rt. ∠s.

Take a second angle four times as great as this angle, and therefore = $\frac{8}{15}$ of two rt. ∠s.

Take a third angle = one of the angles of an equilateral △, and therefore = $\frac{5}{15}$ of two rt. ∠s.

Then describe a △ with its angles equal to these angles, and then inscribe in the given ⊙ a triangle with its angles equal to the angles of this △.

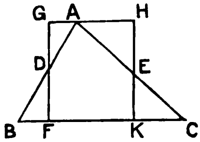


Fig. 317.

1852. I. 42. Let ABC be the given △.

Bisect AB in D, and AC in E.

Draw GDF making ∠ GFC equal to one of the given angles. Through A draw GAH ∥ to BC, and through E draw HEK ∥ to GF.

Then since △ ADG = △ BDF in all respects;

and △ EHA = △ EKC in all respects;

the lines DF, EK divide the triangle in the required manner.

II. 12. Let BC be the base, ABC one of the △s.

Draw AD ⊥ to BC, or BC produced.

Then since (fig. 1) sq. on AB = sqq. on AC, BC with twice rect. BC, CD;

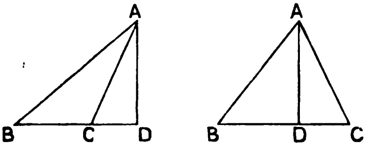


Fig. 1. Fig. 2.

Fig. 318.

∴ since difference of sqq. on AB, AC is constant, and BC is constant,

∴ CD is constant;

∴ the vertex lies in the line drawn from A ⊥ to BC produced.

Similarly (fig. 2), it may be shown that, in the case of ∠ at C being *an acute ∠,* that the vertex lies in the ⊥ on BC from A.

We have taken AB greater than AC, and if we take AC greater than AB, we can show in the same way that the vertex lies in another fixed st. line.

IV. 3. Let ABC, DEF be the $\triangle$s; O, P two of the pts. of contact; R, N, Q three pts. of intersection of the sides.

Then $DO = \frac{1}{2} DE = \frac{1}{2} AC = AP$,
and $NO = NP$,
and $\therefore AN = DN$.
Also, $\angle ANR = \angle DNQ$,
and $\angle NAR = \angle NDQ$,
$\therefore NR = NQ$. (I. 26.)

Similarly, the other sides of the hexagon may be proved to be equal.

When ED is $\parallel$ to BC, $\angle ARN = \angle ANR$;
and $\therefore \angle NRB = \angle RNC$;
and the hexagon is equiangular; but not otherwise.

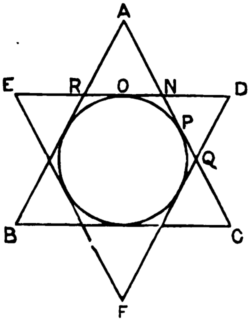


Fig. 319.

1853. I. B. Cor. Let ABC be an isosceles $\triangle$, having $AB = AC$. Draw BDF, CDE as directed.

Then $\triangle BDC$ is evidently isosceles.

Also,
$\angle DBE + \angle BDE + \angle BED =$ two rt. $\angle$s;
and $\angle DBE + 4 \angle CBD =$ two rt. $\angle$s;
and $\angle BDE = 2 \angle CBD$;
$\therefore \angle BED = 2 \angle CBD = \angle BDE$;
$\therefore \triangle BDE$ is isosceles; and so is $\triangle CDF$.

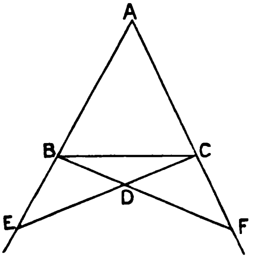


Fig. 320.

I. 29. Let A, D be the pts. and BM the line.
Draw $AC \perp$ to CB.
Make $\angle CAN = \frac{1}{3}$ of a rt. $\angle$.
Then $\angle CNA = \frac{2}{3}$ of a rt. $\angle$.
Draw $DE \parallel$ to BC.
Make $\angle EDF = \frac{2}{3}$ of a rt. $\angle$;
$\therefore \angle OFN = \frac{2}{3}$ of a rt. $\angle$;
$\therefore FON$ is an equilateral $\triangle$.

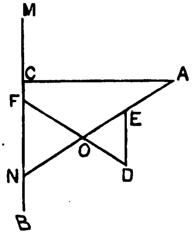


Fig. 321.

II. 11. Since $FA = AH$, and $AB = AD$, and $\angle FAB = \angle DAH$,
$\therefore \angle AFB = \angle AHD$;
$\therefore \angle LHB = \angle AFB$, and $\angle LBH = \angle FBA$,
$\therefore \angle BLH = \angle FAB =$ a rt. $\angle$.

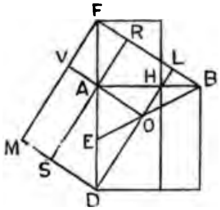


FIG. 322.

Next, $EB = EF$, and $\therefore \angle EBF = \angle EFB$.
$\therefore$ in right-angled $\triangle$s FLD, OLB,
$\because \angle FLD = \angle OLB$, and $\angle DFL = \angle OBL$,
$\therefore \angle FDL = \angle LOB$;
$\therefore \angle EDO = \angle EOD$, and $\therefore EO = ED = EA$,
and $\therefore \angle AOD$ is a rt. $\angle$.
Again, since $EB = EF$, and $EO = EA$,
$\therefore OB = AF = AH$.
$\therefore$ in $\triangle$s AOH, OLB, since $\angle OAH = \angle LOB$,
and $\angle AOH = \angle OLB$, and $AH = OB$;
$\therefore AO = LO$, and $HO = LB$.

Also, $DH = FB$, and $HO = LB$, and $\therefore DO = FL$. Complete the rectangles $DLFM$, $DLRAS$.

Then rect. DL, $LO =$ fig. $RD =$ fig. $VD =$ sq. on DO.

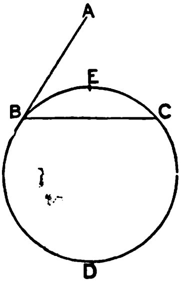


FIG. 323.

III. 32. Let A be the given pt., BCD the circle, 2θ the given angle. Draw AB a tangent to the $\odot$ at B, and make $\angle ABC =$ a rt. angle $- \theta$.

Then angle in segment $BDC =$ a rt. angle $- \theta$;
and angle in segment $BEC =$ a rt. angle $+ \theta$;
$\therefore$ difference of the angles in the segments $= 2\theta$.

Then if in $\odot$ BCD a concentric $\odot$ be described touching BC, and a tangent to the inner $\odot$ be drawn from A, the part of this tangent intercepted by $\odot$ BCD will cut off the required segments.

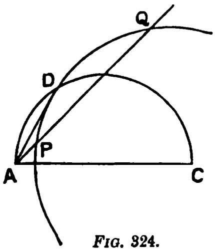


FIG. 324.

III. 36. Let C be the given centre, and A the fixed point in the given line APQ.

Describe a semicircle on AC as diameter, and place in it $AD =$ side of given square.

With centre C and radius CD describe a circle cutting the given line in P, Q. This $\odot$ touches AD, because $\angle ADC$ is a right angle.
$\therefore$ rect. AP, $AQ =$ sq. on AD.

1854. I. 43. Join DB.

Then $KC - KA$
$= OB - OA$,
$= 2(\triangle CKB - \triangle AKD)$,
$= 2(\triangle CNB + \triangle NKB - \triangle AND + \triangle DKN)$,
$= 2\triangle BKD$.

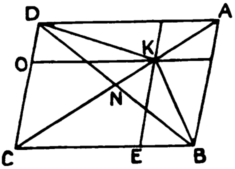


FIG. 325.

II. 11. Let AB be the given line.

Bisect AB in D, and draw $BC = AB$, and $\perp$ to AB.

Join DC, and produce AB to F, so that $DF = DC$.

Then rect. AF, FB with sq. on DB
$=$ sq. on DF, (II. 6.)
$=$ sq. on DC,
$=$ sqq. on CB, BD;
$\therefore$ rect. AF, $FB =$ sq. on CB,
$=$ sq. on AB.

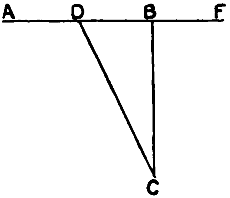


FIG. 326.

III. 22. Let the $\odot$s described about ABP, BCQ meet in R.

Then $\angle PRB =$ supplement of $\angle PAB$,
$= \angle DAB$.
And $\angle QRB =$ supplement of $\angle QCB$,
$= \angle DCB$.
Now sum of $\angle$s DAB, $DCB = 2$ rt. $\angle$s;
$\therefore$ sum of $\angle$s PRB, $QRB = 2$ rt. $\angle$s;
$\therefore$ PRQ is a straight line.

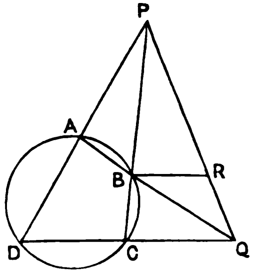


FIG. 327.

IV. 10. Let PQ be the given st. line.

Make $\angle PQO = \angle BAD$ in the Proposition,
and $\angle QPO = \angle BAD$.
Each of these $\angle$s is $\frac{1}{5}$ of two rt. $\angle$s;
$\therefore$ $\angle POQ = \frac{3}{5}$ of two rt. $\angle$s;
$\therefore$ $\angle POQ$ is treble of each of the angles at the base.

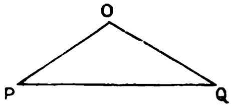


FIG. 328.

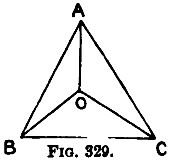


FIG. 329.

1855. I. 20. Let O be the point.
Then sum of OA, OC is greater than AC;
sum of OC, OB is greater than BC;
sum of OB, OA is greater than BA;
$\therefore$ sum of OA, OB, OC is greater than half the sum of AB, BC, CA.

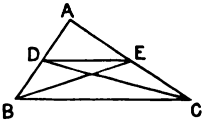


FIG. 330.

I. 47. Let DE be drawn ‖ to BC the hypotenuse of the right-angled $\triangle ABC$. Join BE, CD.
Then sq. on DC = sum of sqq. on DA, AC,
and sq. on BE = sum of sqq. on BA, AE;
$\therefore$ sum of sqq. on DC, BC = sum of sqq. on BA, AC, DA, AE,
= sum of sqq. on BC, DE.

II. 9. See Exercise 41 on p. 119.

III. 27. Let BAC be any one of the $\triangle$s.
Draw BDP, CDO $\perp$s on AC, AB.
Then since $\angle$s at O, P are rt. $\angle$s, $\angle BDC = \angle ODP$,
= supplement of $\angle BAC$,
$\therefore \angle BDC$ is a constant $\angle$.

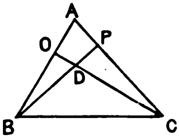


FIG. 331.

$\therefore$ locus of D is the segment of a $\odot$ described on BC, capable of containing $\angle BDC$.
Also, the line bisecting $\angle BDC$ must pass through the centre of the $\odot$ described about $\triangle BDC$, and similarly for the lines bisecting $\angle BDC$ in other positions.

IV. 4. Join DG, EG, and draw GK, GH $\perp$s to AD, AE.

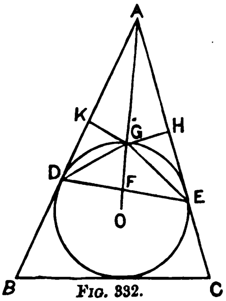


FIG. 332.

Let O be the centre of the $\odot$ inscribed in $\triangle ABC$.
Then $\because AE = AD$, and AF is common, and $\angle DAF = \angle EAF$,
$\therefore GF$ bisects DE at rt. $\angle$s;
$\therefore GD = GE$, and $\angle GEF = \angle GDF$,
$\therefore \angle GDK = \angle GEH$, and $\therefore GK = GH$.
Also, $\angle HEG = \angle GDE$, (III. 32.)
$= \angle GEF$;
$\therefore GF = GH$.
Hence a $\odot$ described with centre G and distance GF will touch the sides of $\triangle ADE$.

1856. I. 34. Let *ABCD* be a rhombus, whose diameters *AC*, *BD* intersect in *O*. Then ∠s at *O* are rt. ∠s.

Then area of rhombus = rect. *AO*, *BD*.

Now let *ABCD* be any ▱ not a rhombus, having its diagonals equal to the diagonals of the rhombus.

Then, if *AP* be drawn ⊥ to *BD*,

area of ▱ *ABCD* = rect. *AP*, *BD* ;

and *AP* is less than *AO*, ∴ area of the ▱ is less than area of the rhombus.

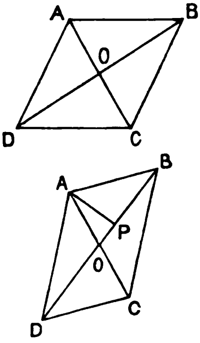


FIG. 333.

II. 12. Sum of sqq. on *AC*, *CD* = twice sum of sqq. on *CB*, *BA* (p. 91, Ex.);

∴ sum of sqq. on *AB*, *CD* = twice sum of sqq. on *CB*, *BA* ;

∴ sq. on *CD* = twice sq. on *CB* with sq. on *BA*.

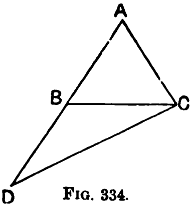


FIG. 334.

IV. 15. Let *ABC* be an equilateral △ inscribed in the ⊙ of which *O* is the centre.

Produce *BO* to meet the Oce in *D*, and join *AD*, *DC*.

Then ∠ *AOD* = twice ∠ *OAB*,

= $\frac{1}{3}$ of two rt. ∠s.

∴ *AD* is the side of a regular hexagon inscribed in the ⊙;

and since △ *AOD* is clearly equilateral, *AD* = radius of circle.

Similarly, *DC* may be shown to be a side of this hexagon.

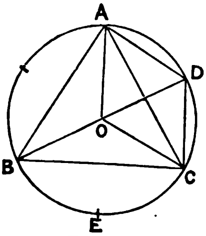


FIG. 335.

And if *AO*, *CO* produced meet the Oce in *E*, *F* ; *CE*, *EB*, *BF*, *FA* are sides of the hexagon.

Also area of hexagon = six times △ *AOD*,

= six times △ *AOB*,

= twice △ *ABC*.

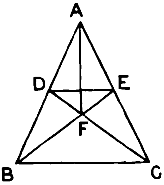


FIG. 336.

1857. I. 35. Sum of $\triangle$s EFC, $EFD = \triangle EDC$ $=\frac{1}{2}\triangle ADC=\frac{1}{4}\triangle ABC$;

sum of $\triangle$s DFB, $EFD = \triangle DEB$ $=\frac{1}{2}\triangle ABE=\frac{1}{4}\triangle ABC$;

$\therefore \triangle EFC + \triangle DFB + 2\triangle EFD=\frac{1}{2}\triangle ABC$;

$\therefore \triangle AFE + \triangle AFD + 2\triangle EFD=\frac{1}{2}\triangle ABC$;

$\therefore \triangle ADE + 3\triangle EFD=\frac{1}{2}\triangle ABC$.

Also, $\triangle ADE=\frac{1}{4}\triangle ABC$;

$\therefore 3\triangle EFD=\frac{1}{4}\triangle ABC$;

$\therefore \triangle ADE=3\triangle EFD$.

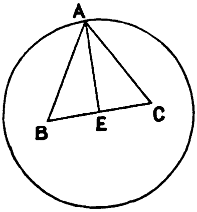


FIG. 337.

II. 13. Let ABC be the $\triangle$, E the centre of the $\odot$, which is of constant radius.

Then since sum of sqq. on BA, AC=twice sum of sqq. on AE, BE; (p. 91, Ex.)

$\therefore$ the sum of sqq. on BA, AC is a constant.

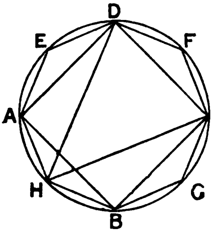


FIG. 338.

III. 22. Let $ABCD$ be the quadrilateral, and take E, F, G, H pts. in the exterior segments. Complete the figure as in the diagram.

Then sum of $\angle$s AED, AHD=two rt. $\angle$s (III. 22.)

and sum of $\angle$s DHC, DFC=two rt. $\angle$s (III. 22.)

and sum of $\angle$s CHB, CGB=two rt. $\angle$s (III. 22.)

$\therefore$ sum of $\angle$s AED, AHB, DFC, CGB=six rt. $\angle$s.

IV. 4. Let ABC be the $\triangle$; AD, BE, CF the $\perp$s;

d_1, d_2 the diameters of $\odot$s inscribed in $\triangle$s ABD, ADC;

d_3, d_4 the diameters of $\odot$s inscribed in $\triangle$s BEC, BEA;

d_5, d_6 the diameters of $\odot$s inscribed in $\triangle$s ACF, BCF.

Then by Ex. 2 on p. 183,

$$AB+d_1=AD+BD,$$
$$AC+d_2=AD+DC,$$
$$BC+d_3=BE+CE,$$
$$AB+d_4=BE+EA,$$
$$AC+d_5=CF+FA,$$
$$BC+d_6=CF+FB;$$

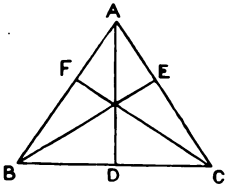


FIG. 339.

∴ adding, we obtain

sum of diameters + twice sum of sides = twice sum of $\perp$ s + sum of sides;

∴ sum of diameters + sum of sides = twice sum of $\perp$ s.

1858. I. 28. The rider has been already explained on p. 47.

II. 7. Let AB be the given line.

Draw $BD \perp$ to AB, and equal to AB.

Join AD. Produce AB to C, so that $AC=AD$.

Then sqq. on AC, $BC=2$ rect. AC, CB with sq. on AB;

∴ sqq. on AB, BD, $BC=2$ rect. AC, CB with sq. on BD;

∴ sqq. on AB, $BC=2$ rect. AC, CB.

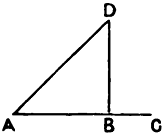


FIG. 340.

III. 19. Let A be the centre of the given $\odot$, B the given pt. in the st. line CD.

Draw $BE \perp$ to CD. Join BA, and produce it to P so that rect. BA, AP = sq. on radius of given $\odot$.

Then P is known, and it is a point in the ○ce of the $\odot$ which has to be described. Bisect BP in O, and draw $OQ \perp$ to BP, and meeting BE in Q, then Q will be the centre of the $\odot$ passing through P and touching CD in B.

Also, since rect. BA, AP = sq. on radius of original $\odot$, the common chord of the two $\odot$s will evidently pass through A, and the new $\odot$ will therefore bisect the original $\odot$.

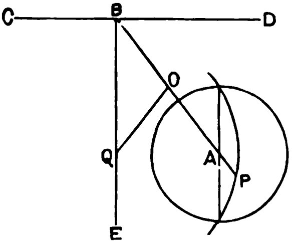


FIG. 341.

1859. I. 41. Let $ABCD$ be the given ▱.

Take EC = one-third of BC,

FC = one-third of DC,

and join AE, AF, and draw $EG \parallel$ to AB.

Then $\triangle\ ABE=\frac{1}{2}$ ▱ AE,

$=\frac{1}{3}$ ▱ $ABCD$;

and similarly $\triangle\ ADF=\frac{1}{3}$ ▱ $ABCD$;

∴ *quadrilateral* $AECF=\frac{1}{3}$ ▱ $ABCD$.

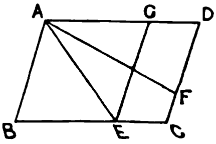


FIG. 342.

II. 13. See Exercise 10 on p. 94.

III. 31. Let ABC, ADE be the equal ⊙s.
Draw the chords AB, AD, at rt. $\angle$s to each other.
Draw $EOAFC$ passing through the centres O, F.

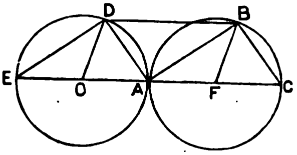


FIG. 343.

Then $\angle$s ABC, ADE are rt. $\angle$s, (III. 31.)

and $\angle DAO + \angle BAC =$ a rt. $\angle$,

$= \angle BCA + \angle BAC$,

$\therefore \angle DAO = \angle BCA$, and $\therefore DA$ is $\parallel$ to BC.

Again, in $\triangle$s EDA, ABC,

$\because \angle EDA = \angle ABC$, and $\angle DAE = \angle BCA$, and $EA = AC$,

$\therefore AD = BC$;

and $\therefore DB$ is $\parallel$ and $=$ to AC (I. 33); $\therefore DB = OF$.

IV. 4. Let BC be the base; BAC the vertical $\angle$; O the centre of one of the escribed ⊙s, touching AC, and the other sides produced.

Join OB, OC, and produce BC to T.

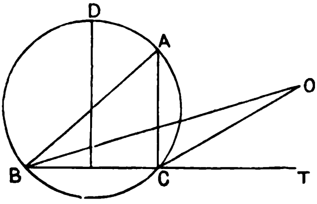


FIG. 344.

Then OB, OC bisect $\angle$s ABC, ACT.

Now $\angle BOC = \angle OCT - \angle OBC$,

$= \frac{1}{2} \angle ACT - \frac{1}{2} \angle ABC$

$= \frac{1}{2} \angle BAC$.

The locus is therefore the segment of a ⊙ passing through B and C, and having $\frac{1}{2} \angle BAC$ as the angle in it. The angle subtended by BC at the centre of this ⊙ being $= \angle BAC$, the centre must lie on the circumference of the circle described about BAC, and as BC is a chord, the centre must be at D, the point farthest from BC.

Note.—If A move to the other side of BC, the conditions are *different*, and the locus will be the centre of another ⊙.

1860. I. 35. Draw $CP \parallel$ to BD.
Then $\because \angle MBD = \angle MCP$,
and $\angle DMB = \angle PMC$,
and $BM = MC$, $\therefore CP = BD$.
Now $\angle AED = \angle ADE$;
$\therefore \angle CEP = \angle CPE$, and $\therefore CE = CP = BD$.

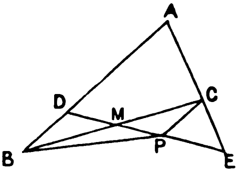
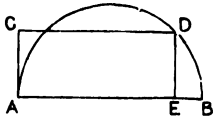


FIG. 345.

II. 14. (1) Let AB be the sum of the sides.
On AB describe a semicircle ADB.
Draw $AC \perp$ to AB and equal to a side of the given square.
Draw $CD \parallel$ to AB, and $DE \perp$ to AB.
Then rect. AE, EB = sq. on ED,
= sq. on CA,
and sum of AE, $EB = AB$.



(2) Let AB be the difference of the sides.
On AB as diameter describe a ⊙ $AEBD$.
Draw $AC \perp$ to AB and equal to a side of the given square.
Draw $CEOD$ through O the centre.
Then rect. CD, CE = sq. on AC;
and difference of CD, $CE = AB$.

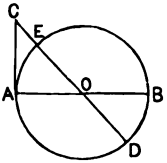


FIG. 346.

III. 36. Join AO, and produce it to meet DE in F.
Then since rect. AC, AE = rect. AB, AD,
a ⊙ may be described about $CBDE$;
$\therefore \angle ABC = \angle AEF$.
Join CQ. Then $\because \angle ABC = \angle AQC$,
$\therefore \angle AEF = \angle AQC$;
$\therefore$ a ⊙ may be described about $CEFQ$;
$\therefore \angle QCE + \angle QFE$ = two rt. $\angle$s;
and $\angle QCE$ is a rt. $\angle$, since AQ is a diameter;
$\therefore \angle QFE$ is a rt. $\angle$.

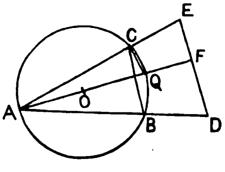


FIG. 347.

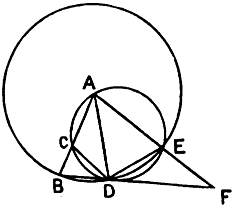


FIG. 348.

IV. 10. Take the diagram in Euclid IV. 10, and complete it as directed.

Then ∠ *AED*=supplement of ∠ *ACD*,
= ∠ *BCD* = ∠ *ABD*;
∴ ∠ *ADE* = ∠ *AED* = ∠ *ABD* = ∠ *ADB*;
∴ third ∠ *EAD*=third ∠ *BAD*.
Then ∠ *BAF* = 2 ∠ *BAD* = ∠ *ABD*;
and ∠ *ABF* = 2 ∠ *BAD* = ∠ *ADB*;
∴ ∠ *AFB* = ∠ *BAD*.

1861. I. 32. Let *EF* be the given st. line.
Draw *FP* ∥ to *CB*, making an acute ∠ with *EF*.
Make ∠ *NCB* = ∠ *EFP*, *CN* meeting *AB* in *N*.
Draw *QNM* ∥ to *EF*.
Then ∵ ∠ *NCM* = ∠ *EFP* = ∠ *NMC*,
∴ *CN*=*NM*.

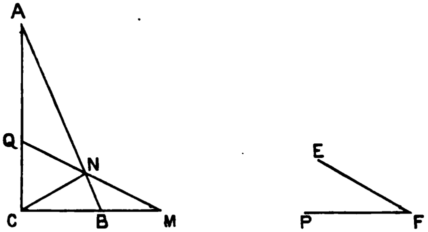


FIG. 349.

Also, ∠ *QCN*=complement of ∠ *NCM*,
=complement of ∠ *NMC*,
= ∠ *CQN*,
∴ *QN*=*NC*=*NM*.

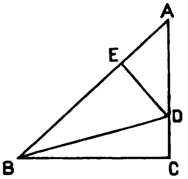


FIG. 350.

II. 13. Sq. on *BD*=sqq. on *BA*, *AD* diminished by 2 rect. *BA*, *AE*;

and sq. on *BD*=sqq. on *BA*, *AD* diminished by 2 rect. *AD*, *AC*;

∴ 2 rect. *BA*, *AE*=2 rect. *AD*, *AC*;
∴ rect. *BA*, *AE*=rect. *AD*, *AC*.

III. 32. Let the tangents meet in T.

$\angle TFE = \angle FCA$, (III. 22.)
$=$ supplement of $\angle FBA$,
$=$ complement of $\angle CBF$,
$=$ complement of $\angle CAF$.

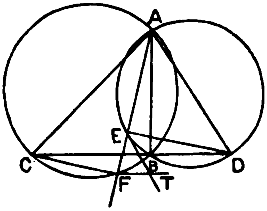


FIG. 351.

$\angle TEF = \angle EDA$, (III. 22.)
$=$ complement of ($\angle EDB + \angle DAB$),
$=$ complement of ($\angle EAB + \angle DAB$),
$=$ complement of $\angle DAF$,
$=$ complement of $\angle CAF$.
$\therefore \angle TFE = \angle TEF$;
$\therefore TE = TF$;

and these tangents being equal, T must be on the common chord of the ⊙s, *i.e.* on AB produced. (See p. 171, Ex. 23.)

The proof is similar when FEA bisects the exterior angle between CA and DA.

IV. 4. The first part of this rider has been proved in Ex. 4 to IV. 4. Let the inscribed ⊙ touch BC in D, and the escribed ⊙ touch BC in E.

Then, from the properties of tangents,

$2AB + 2CD =$ sum of sides of △;
$2AB + 2BE =$ sum of sides of △;
$\therefore CD = BE$;
$\therefore ED = BD - BE$,
$= BD - DC$,
$= BF - CH$,
$= AB - AC$.

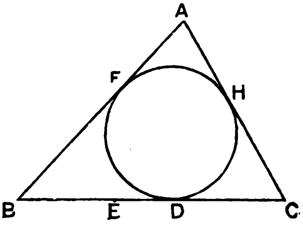


FIG. 352.

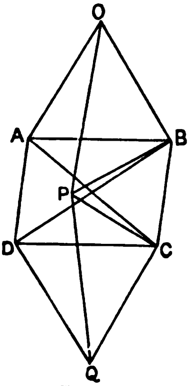


FIG. 353.

1862. I. 4. Let P, O, Q be the vertices of the equilateral $\triangle$s.

Then $\angle PCQ = \angle PCD + \angle QCD$,
$= \angle PCD + \angle PCB$,
$= \angle BCD$;
also $PC = BC$, and $CQ = CD$;
$\therefore PQ = DB$. (I. 4.)

Again, $\angle OBP = \angle OBA + \angle ABP$,
$= \angle PBC + \angle ABP$,
$= \angle ABC$;
and $OB = AB$, and $BP = BC$;
$\therefore OP = AC$. (I. 4.)

II. 10. Let AB be the given line.

Take AE, in AB produced, such that sq. on $AE = 3$ sq. on AB; cut off $AC = BE$.

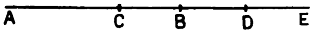


FIG. 354.

Then $CE = AB$.
Make $CD = AC$; then $CB = DE$.
Now since AD is bisected in C and produced to E,
sum of sqq. on AE, $DE =$ twice sum of sqq. on AC, CE;
$\therefore$ 3 sq. on $AB +$ sq. on $CB = 2$ sq. on $AC + 2$ sq. on AB;
$\therefore$ sq. on $AB +$ sq. on $CB = 2$ sq. on AC.

III. 28. Let $AB'C'$ be the new position of the $\triangle ABC$ when it has been turned about A.

Let the two positions of the base produced intersect in D. Join AD.

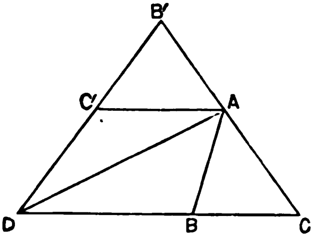


FIG. 355.

Then
$\angle AC'D =$ supplement of $\angle AC'B'$,
$=$ supplement of $\angle ACD$;
$\therefore$ a circle can be described about $ACDC'$; and since chord $AC =$ chord AC',
$\therefore \angle ADC' = \angle ADC$.

IV. 10. In arc AED of the smaller ⊙ take any pt. E.

Then $\angle AED$ = supplement of $\angle ACD$,

$= \angle BCD$,

$= \angle ABD$;

∴ chord AD subtends equal angles in the circle AED and in the circle described about △ ABD;

∴ these ⊙s are equal.

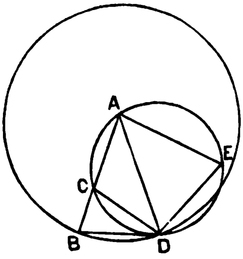


FIG. 356.

1863. I. 47. Draw the ⊥s from A, B, C to opposite sides meeting those sides in α, β, γ; α_1, β_1, γ_1.

Then sqq. on $B\alpha$, $A\alpha$, $C\beta$, $B\beta$, $A\gamma$, $C\gamma$,

= sqq. on AB, BC, CA,

= sqq. on $C\alpha$, $A\alpha$, $B\beta$, $A\beta$, $B\gamma$, $C\gamma$.

And taking away common squares,

sqq. on $B\alpha$, $C\beta$, $A\gamma$ = sqq. on $C\alpha$, $A\beta$, $B\gamma$;

∴ sqq. on $B_1\alpha_1$, $A\alpha_1$; $C_2\beta_1$, $B\beta_1$; $A_3\gamma_1$, $C\gamma$,

= sqq. on $C_1\alpha_1$, $A\alpha_1$; $A_2\beta_1$, $B\beta_1$; $B_3\gamma_1$, $C\gamma_1$;

∴ sqq. on AB_1, BC_2, CA_3 = sqq. on AC_1, BA_2, CB_3.

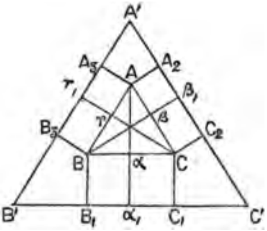


FIG. 357.

II. 11. Let AB be the line to be divided.

On AB describe a square $ABDC$, produce CA to E so that AE = the other given line.

Complete rectangle $AEMB$, and make rectangle $ECKG = AEMB$. Let GK cut AB in H.

Then, taking away rectangle EH,

$AK = GB$;

or, rect. AB, AH = rect. HB, AE:

∴ H is the point required.

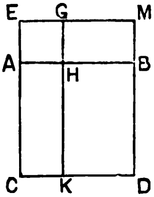


FIG. 358.

III. 28. Solved before as Exercise 31, on p. 172.

1864. I. 38. Let $ABCD$ be the quadrilateral, having $AB \parallel$ to CD. Join AC, BD, intersecting in O; bisect AB in E; join EO. Then EO produced shall pass through the middle pt. of DC.

For, if not, let F be the middle pt. of DC, such that EPF is a st. line, and join FO.

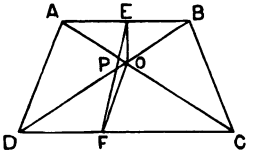


FIG. 359.

Then $\triangle DAB = \triangle CAB$,

and $\triangle OAB$ is common,

$\therefore \triangle DAO = \triangle CBO$.

Also $\triangle EAO = \triangle EBO$,

and $\triangle FDO = \triangle FCO$;

$\therefore$ figure $DAEOF$ = figure $CBEOF$;

$\therefore$ figure $DAEOF$ = half of figure $ABCD$.

But $\triangle FEA = \triangle FEB$, and $\triangle AFD = \triangle BFC$;

$\therefore$ figure $DAEPF$ = figure $CBEPF$;

$\therefore$ figure $DAEPF$ = half of the figure $ABCD$,

= figure $DAEOF$, which is absurd.

Hence F is not the middle pt. of DC.

Similarly, it may be shown that no point, except one in EO produced, is the middle pt. of DC.

II. 14. Bisect the given st. line AB in P, and divide it in Q, so that rect. AB, BQ = given rectilineal figure. (I. 45 and II. 14.)

Take $BD = PQ$, and divide AD in C (if possible), so that rect. AC, CD = given rectilineal figure (II. 14); then C and D shall be the points of section required.

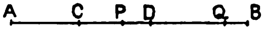


FIG. 360.

For rect. AC, CD = given figure = rect. AB, BQ = rect. AB, DP.

And sq. on AD = sqq. on AC, CD with 2 rect. AC, CD,

= sqq. on AC, CD with 2 rect. AB, DP,

= sqq. on AC, CD with 4 rect. AP, DP;

also, sq. on AD = sqq. on AP, PD with 2 rect. AP, DP,

= sqq. on BP, PD with 2 rect. AP, DP,

= sq. on BD with 4 rect. AP, DP; (*Eucl.* II. 7.)

$\therefore$ sqq. on AC, CD = sq. on BD.

III. 36. Let *C* be the centre of the circle, *ABCD* the diameter. It is evident that *PS*, *QR*, intersect on some point *O* in *AD*.

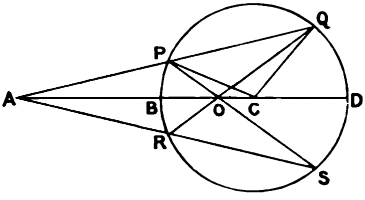


FIG. 361.

Also $\angle PCA = \frac{1}{2} \angle PCR = \angle PQR$;

$\therefore$ *POCQ* can be circumscribed by a ⊙;

$\therefore$ rect. *AO*, *AC* = rect. *AP*, *AQ*,

= rect. *AB*, *AD*;

$\therefore$ *O* is a fixed point.

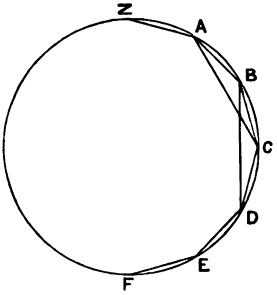


FIG. 362.

IV. 11. Let *Z*, *A*, *B*, *C* . . . be consecutive angles of the figure.

Join *ZA*, *AB*, *BC*, *CD*, *AC*, *BD*.

Then since $\angle ABC = \angle BCD$,

$\therefore$ arc *AC* = arc *BD*;

and, taking away the common part *BC*,

arc *AB* = arc *CD*:

and $\therefore$ side *AB* = side *CD*.

Similarly, side *AB* = *CD* = *EF* = . . . = *ZA* = *BC* = . . . the number of sides being *odd*. So all the sides are equal.

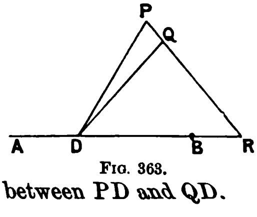


FIG. 363.

1865. I. 20. Let *AB* be the given line, *P* and *Q* the given points. Join *PQ*, and produce it to meet *AB*, or *AB* produced, in *R*. Then shall *PQ* be greater than the difference of any two lines drawn from *P* and *Q* to the same point in *AB*, as *PD*, *QD*.

For sum of *PQ*, *QD* is greater than *PD*;

$\therefore$ *PQ* is greater than the difference between PD and QD.

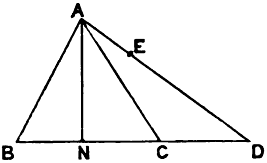


FIG. 364.

II. 12. Draw $AN \perp$ to, and $\therefore$ bisecting, BC.

Then sq. on AD=sqq. on AC, CD, with 2 rect. CD, CN;

$\therefore$ rect. AD, AE with rect. AD, DE =sqq. on AC, CD with 2 rect. CD, CN;

$\therefore$ rect. AD, DE=sq. on CD with rect. CD, BC,
=rect. BD, CD. (II. 3.)

III. 18. Let DC be $\perp$ to AB the line joining the centres A and B. Draw DP, DQ, tangents to the $\odot$s. Join AP, AD, BQ, BD.

Then sq. on PD=sq. on AD − sq. on AP,
sq. on QD=sq. on BD − sq. on BQ;

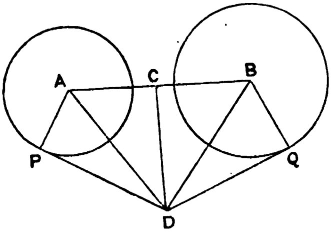


FIG. 365.

$\therefore$ difference of sqq. on PD, QD=difference of sqq. on AD, BD with difference of sqq. on BQ, AP.

Now difference of sqq. on AD, BD=difference of sqq. on AC, BC,
=a constant;
and difference of sqq. on BQ, AP=a constant;
$\therefore$ difference of sqq. on PD, QD is constant.

IV. 5. Let AB be the given side. Then since the centre of the circumscribed $\odot$ is fixed, the angle ACB is of constant magnitude.

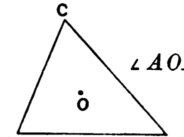


FIG. 366.

$\therefore$ sum of $\angle$s CAB, CBA is constant;
and, if O be the centre of the inscribed $\odot$,
$\angle AOB$=supplement of half the sum of $\angle$s CAB, CBA;
$\therefore$ $\angle AOB$ is constant, and the locus of O is therefore a $\odot$ passing through A and B.

1866. I. 33. Let $AB = CD$, and $\angle CAB = \angle BDC$.

Then since sides AB, BC in $\triangle ABC$ are equal to sides BC, CD in the other, and $\angle$s ACB, CBD are both acute, it may be shown as in I. Prop. E, p. 43, that $AC = BD$, and $\angle ABC = \angle BCD$, and $\angle ACB = \angle CBD$, and $\therefore ABDC$ is a ▱.

But if $\angle$s at A and D be acute, $\angle$s ACB, CBD are not necessarily *both* acute or *both* obtuse, and $\therefore$ I. Prop. E will not apply.

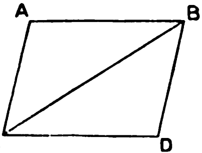


FIG. 367.

II. 9. We have to prove that

$$Ag + DE = 2\,(af + Cd),$$
$$\text{or, } 2Ac + DE = af + Cd,$$
$$\text{or, } Ac + \triangle DBE = \triangle eac + \triangle cCD.$$

Draw $ckK \perp$ to ED.

Then $Ac - \triangle cCD + \triangle DBE$

$$= Ac - \triangle cCk + \triangle AkK,$$
$$= \triangle aKc,$$
$$= \triangle eac.$$

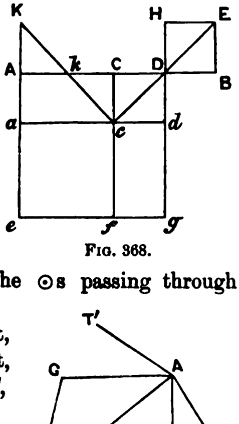


FIG. 368.

III. 32. Let AT, AT' be tangents to the ⊙s passing through ABD, ABC.

Then $\angle TAD = \angle ABD$ in alternate segment,

and $\angle T'AC = \angle ABC$ in alternate segment,

$\therefore \angle T'AT = \angle CAD + \angle ABD + \angle ABC,$

$= \angle CAD + \angle CBD.$

So angle between tangents to the ⊙s passing through ACD, CBD,

$= \angle ACB + \angle ADB,$

= four rt. $\angle$s diminished by sum of $\angle$s CAD, CBD.

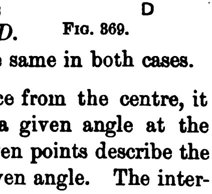


FIG. 369.

Hence the acute angles between them are the same in both cases.

IV. 2. As the third side is at a given distance from the centre, it is of a given length, and therefore subtends a given angle at the circumference. On the line joining the two given points describe the segment of a circle capable of containing the given angle. The intersection of this segment with the given circle will give the vertex of the triangle.

Note.—There will generally be two solutions, or none at all; but in the particular case when the segment touches the circle, only one solution.

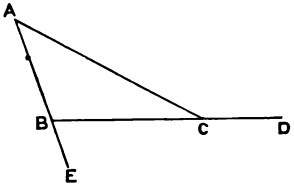


FIG. 370.

1867. I. 16. Let *CBE*, *ACD* be any two exterior ∠s of the △*ABC*.

Then ∵ ∠ *ACD* is greater than ∠ *ABC*,

sum of ∠s *ACD*, *CBE* is greater than sum of ∠s *CBE*, *ABC* ;

∴ sum of ∠s *ACD*, *CBE* is greater than two rt. ∠s.

I. 43. The greatest value which each complement can have is one-fourth of the parallelogram, when $AE = ED$.

II. 11. Solved before. See Riders in 1862.

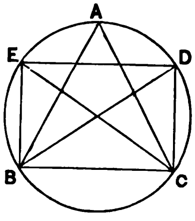


FIG. 371 (1).

III. 22. Let *DB*, *EC* be equal chords bisecting two ∠s in the △*ABC* inscribed in a ⊙.

Now ∠ *DCB* = ∠ *EBC*, subtended by equal chords ;

∴ sum of ∠s *EDC*, *DCB*=two rt. ∠s ;

∴ *ED* is ∥ to *BC* ;

∴ ∠ *DBC* = ∠ *EDB*,

= ∠ *ECB* ;

and ∴ ∠ *ABC*=2 ∠ *DBC*=2 ∠ *ECB* = ∠ *ACB*.

Next, if the bisectors be on opposite sides of the centre,

∠ *EBC* = ∠ *BAD*, subtended by equal chords.

But ∠ *EBC* = ∠ *ABC* + ∠ *EBA*,

= ∠ *ABC* + ∠ *ECA*,

= ∠ *ABC* + $\frac{1}{2}$ ∠ *ACB* ;

and ∠ *BAD*= ∠ *BAC*+ ∠ *CAD*,

= ∠ *BAC* + ∠ *CBD*,

= ∠ *BAC* + $\frac{1}{2}$ ∠ *ABC* ;

∴ ∠ *BAC* + $\frac{1}{2}$ ∠ *ABC*= ∠ *ABC* + $\frac{1}{2}$ ∠ *ACB*;

∴ ∠ *BAC*=$\frac{1}{2}$ (∠ *ABC*+ ∠ *ACB*),

and ∴ ∠ *BAC*=two-thirds of a rt. ∠.

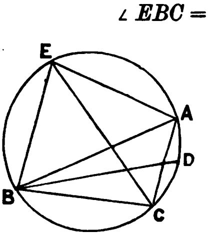


FIG. 371 (2).

1868. I. 41. Through Q draw $RQS \parallel$ to NM.

Then $\triangle BQD = \triangle ODQ + \triangle OQB$,
$= \frac{1}{2}\square NQ + \frac{1}{2}\square OS$,
$= \frac{1}{2}\square NRSM$.
And $\triangle PQD = \triangle PDC - \triangle QDC$,
$= \frac{1}{2}\square NC - \frac{1}{2}\square RC$,
$= \frac{1}{2}\square NRSM$.
$\therefore \triangle BQD = \triangle PQD$;
$\therefore PB$ is $\parallel$ to QD.

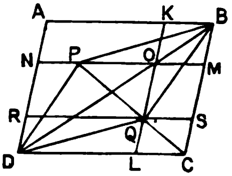


Fig. 372.

II. 12. Sq. on AC=sqq. on AD, DC+2 rect. CD, DK.
And sqq. on AB, AC=2 sqq. on AD, DC. (II. 13, Ex.)
$\therefore$ 2 sq. on AC=sqq. on AB, AC+2 rect. BC, DK;
$\therefore$ sq. on AC=sq. on AB+2 rect. BC, DK;
and sq. on BC=sq. on AB+2 rect. AC, EL;
and sq. on BC=sq. on AC − 2 rect. AB, FM;

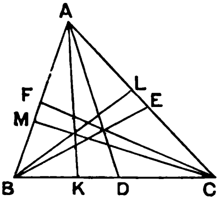


Fig. 373.

$\therefore$ 2 rect. BC, DK=sq. on AC − sq. on AB;
2 rect. AB, FM=sq. on AC − sq. on BC;
2 rect. AC, EL=sq. on BC − sq. on AB;
$\therefore$ 2 rect. BC, DK=2 rect. AB, FM+2 rect. AC, EL;
$\therefore$ rect. BC, DK=rect. AB, FM+rect. AC, EL.

III. 35. Let P be the point, C the centre, ACA' a diameter through P, NM the side of the given square, and describe a semicircle on NM, and place NO in it, NO being the side of a square=sq. on CA − sq. on CP.

With centre P and radius OM, describe a ⊙ cutting the original ⊙ in Q.

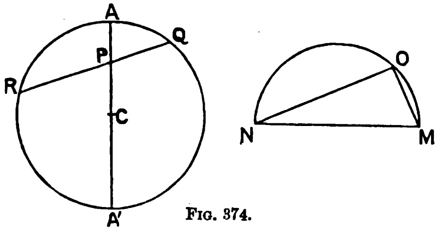


FIG. 374.

Then rect. RQ, PQ = rect. RP, PQ + sq. on PQ,
= rect. AP, PA' + sq. on PQ,
= sq. on CA − sq. on CP + sq. on PQ,
= sq. on ON + sq. on OM,
= sq. on NM.

The limits are deducible from the fact that OM must be greater than PA and less than PA'.

IV. 4. Let O be the centre of the ⊙ inscribed in △ ABC; then OA, OB, OC bisect the angles at A, B, C.

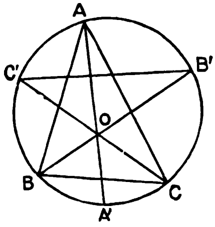


FIG. 375.

$\therefore \angle AOC' = \angle OAC + \angle OCA$,
$= \frac{1}{2} \angle BAC + \frac{1}{2} \angle ACB$;
and $\angle B'C'O = \angle B'BC$,
$= \frac{1}{2} \angle ABC$;
$\therefore \angle AOC' + \angle B'C'O$
$= \frac{1}{2}(\angle BAC + \angle ACB + \angle ABC)$,
= a rt. angle :
$\therefore AA'$ is ⊥ to $B'C'$;
$\therefore O$ is the centre of ⊥ s of △ $A'B'C'$.

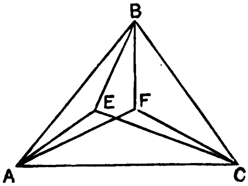


FIG. 376.

1869. I. 40. Subtracting the sums of △s AEB, BEC, and AFB, BFC from the whole △ ABC,

△ AEC = △ AFC;
$\therefore EF$ is ∥ to AC.

If the points E, F, or either, lie on the opposite side of AB, the △s AEB, AFB are (both or one) negative, and similarly for the other sides of △ ABC.

II. 11. In the diagram of the Proposition

$EF = EB$, and $\therefore EF$ is less than $EA + AB$;
$\therefore AF$ is less than AB;
$\therefore AH$ is less than AB;
$\therefore H$ lies between A and B.

III. 33. Let ABC be the △ inscribed in the ⊙ PQR.

Let the segments APC, ARB, when folded meet in O.

Then $\angle AOC = \angle$ in segment APC,
$=$ supplement of $\angle ABC$;
so $\angle AOB =$ supplement of $\angle ACB$;
$\therefore \angle BOC =$ supplement of $\angle BAC$,
$= \angle$ in segment BQC;

so that when BQC is folded it must pass through O.

Should one $\angle$ be obtuse, the circumferences of two of the segments when produced will intersect on the third.

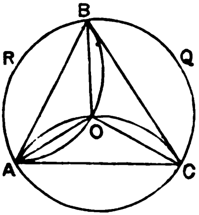


FIG. 377.

IV. 11. Let O be the centre of the regular pentagon $ABCDE$.

Produce CB, EA to T, T'.

Then $\angle TBA = \angle T'AB = \frac{1}{5}$ of 4 rt. $\angle$s,
$= \angle AOB$;
$\therefore$ the circles all pass through O.

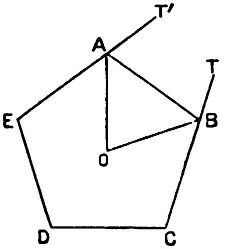


FIG. 378.

1870. I. 26. Take in AB a part $BN = BE$,
and $\therefore AN = EC$.

Then $\angle FEC + \angle AEB =$ a rt. $\angle$,
$= \angle AEB + \angle BAE$;
$\therefore \angle FEC = \angle NAE$.

Also $\angle ANE = \angle NBE + \angle NEB$,
$= \frac{3}{2}$ of a rt. $\angle$,
$= \angle ECF$.

Now since $\angle FEC = \angle NAE$,
and $\angle ECF = \angle ANE$, and $AN = EC$,
$\therefore FE = AE$.

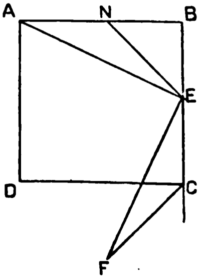


FIG. 379.

II. 9. By the Proposition

sqq. on DE, $EB = 2$ sqq. on DO, OE;
sqq. on AE, $EC = 2$ sqq. on AP, EP.

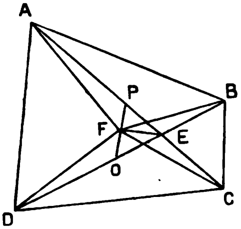


FIG. 380.

∴ sum of sqq. from E to the angular pts.,
$= 2$ sq. on $DO + 2$ sq. on $AP + 2$ (sq. on $OE +$ sq. on EP),
$= 2$ sq. on $DO + 2$ sq. on $AP + 4$ (sq. on $EF +$ sq. on FP),
(By Ex. on p. 91.)
$= 2$ sq. on $DO + 2$ sq. on $AP + 4$ sq. on $EF + 2$ sq. on $FP + 2$ sq. on FO,
$= 2$ (sq. on $DO +$ sq. on FO) $+ 2$ (sq. on $AP +$ sq. on FP) $+ 4$ sq. on EF,
$=$ sum of sqq. on FD, $FB +$ sum of sqq. on FA, $FC + 4$ sq. on EF,
$=$ sum of sqq. from F to the angular pts. $+ 4$ sq. on EF.

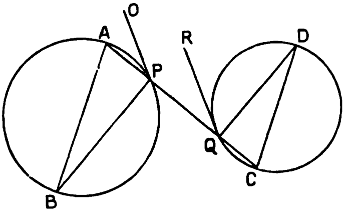


FIG. 381.

III. 32. Let OP be the tangent at P, and RQ the tangent at Q.

Then
$\angle OPA = \angle ABP$,
$=$ complement of $\angle PAB$,
$=$ complement of $\angle QCD$,
$= \angle QDC$,
$= \angle RQP$;
∴ OP is || to RQ.

IV. 10. Each of the angles at the base of the triangle ABD in the diagram of the Proposition is $\frac{2}{5}$ of two rt. angles.

From the centre of the given circle draw five radii inclined to each other successively at angles $= \frac{2}{5}$ of two rt. angles.

The tangents at the extremities of these radii will form the circumscribing regular pentagon.

1871. I. 38. $\triangle CA'B' = \triangle AA'C + \triangle AA'B'$,
$= \triangle AA'C + \triangle ABA'$,
$= \triangle ABC$.
$\triangle BC'A' = \triangle ABA' + \triangle AA'C'$,
$= \triangle ABA' + \triangle AA'C$,
$= \triangle ABC$.
$\triangle AB'C' = \triangle BB'C' - \triangle ABB'$,
$= \triangle BB'C - \triangle ABB'$,
$= \triangle ABC$.

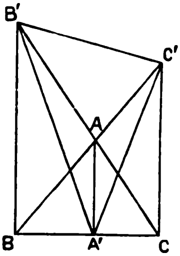


FIG. 382.

II. 10. Let AC be the given st. line.
Produce it to B, so that $CB = AC$.
Draw $CE \perp$ to AB, and $= AC$.
Join AE, EB and produce AB to D, so that $BD = EB$.
Then sq. on AD + sq. on BD = 2 sq. on AC + 2 sq. on CD.
But sq. on BD = sq. on EB.
= 2 sq. on AC.
$\therefore$ sq. on AD = 2 sq. on CD.

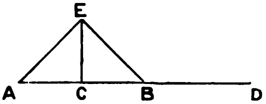


FIG. 383.

III. 32. Let $ABCD$ be the quadrilateral.
Let QT, QT' be the tangents at Q.
Then $\angle TQB = \angle PCQ$,
$= \angle DAB$;
$\therefore QT$ is $\parallel$ to PD, and so also QT' is $\parallel$ to CP.
Again, let PR, PR' be the tangents at P.
Then PR', PR are $\parallel$ to QD and QA respectively.
$\therefore$ the new figure formed is placed precisely similar with respect to QP as the old one ($ABCD$) is with respect to PQ, and the figures are equal in all respects; and any line joining similar points with respect to the figures must bisect PQ, and $\therefore$ in particular the line joining the centres spoken of must bisect PQ.

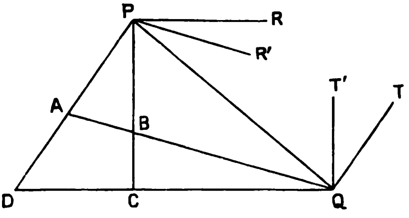


FIG. 384.

IV. 5. Let ABC be the △; O the centre of ⊥s.

Then since ∠s at c and b are rt. ∠s,

$\therefore \angle cOb$ is supplement of $\angle cAb$,

$\therefore \angle BOC$ is supplement of $\angle cAb$.

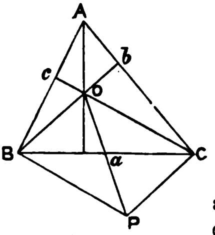


Fig. 385.

Let a be the middle pt. of BC, and produce Oa to P, so that $aP = aO$.

Then $BOCP$ is a parallelogram,

and $\angle BPC = \angle BOC$,

$= $ supplement of $\angle BAC$.

$\therefore P$ lies on the ○ce of the ⊙ about ABC, and since O is fixed, a must lie on the ○ce of a ⊙ of half the radius of the ⊙ circumscribing ABC, O being their centre of similitude.

1872. I. 47. Draw AP, AQ ⊥s to EB, CD.

Then since △s EAB, CAD are equal, and their bases EB, CD are equal,

$\therefore$ their altitudes AP, AQ are equal.

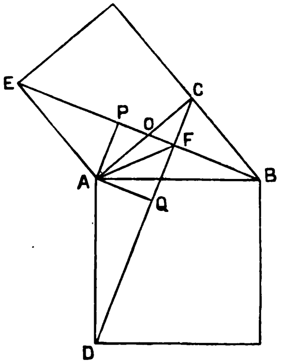


Fig. 386.

Again $\angle AEO = \angle FCO$, and $\angle EOA = \angle COF$,

$\therefore \angle OFC = \angle OAE =$ a rt. angle;

$\therefore PAQF$ is a square;

and $\therefore AF$ bisects $\angle EFD$.

III. 22. Through B draw RBD, meeting the ⊙s in R, D.

Then $\angle P'DB$ = supplement of $\angle BAP'$,

$= \angle BAP$,

$= \angle Q'AB$;

$\therefore P'B = Q'B$.

Similarly, $PB = QB$.

Also, $\angle PBQ = \angle PAQ$,

$= \angle P'AQ'$,

$= \angle P'BQ'$;

and $\therefore \angle PBP' = \angle QBQ'$.

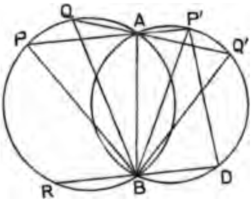


Fig. 387.

Hence in △s PBP', QBQ',

$\because PB = QB$, and $P'B = Q'B$, and $\angle PBP' = \angle QBQ'$,

$\therefore PP' = QQ'$.

IV. 4. Let A be the given angular point, E the centre of ⊙ inscribed in the △.

Join AE and produce it to cut the circumscribing ⊙ in D.

With D as centre, and DE as radius, describe a ⊙ cutting the given ⊙ in B and C.

Then shall ABC be the required △.

For DA bisected $\angle BAC$, since $DB = DC$;

$\therefore$ if EC bisects $\angle ACB$, E will be the centre of the ⊙ inscribed in △ ABC.

Now $\angle DCB = \angle DAB$,

$= \frac{1}{2} \angle BAC$.

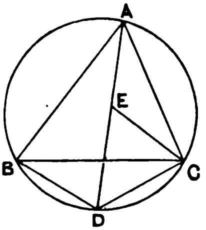


Fig. 388.

Also in the isosceles △ EDC,

sum of $\angle$s DEC, ECD = supplement of $\angle EDC$,

= supplement of $\angle ABC$,

= sum of $\angle$s BAC, ACB;

$\therefore \angle ECD = \frac{1}{2}$ sum of $\angle$s BAC, ACB;

and $\angle DCB = \frac{1}{2} \angle BAC$,

$\therefore \angle ECB = \frac{1}{2} \angle ACB$.

BOOK VI.

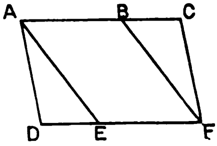


FIG. 389.

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EXERCISE 1. Since the altitudes of the △s *ADE*, *FCB* are equal,

$$\therefore \triangle ADE : \triangle FBC = DE : BC.$$

Ex. 2. Let *P* be a point in *BD*, a diagonal of the ▱ *ABCD*, and join *PA*, *PC*.

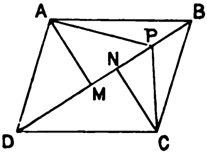

FIG. 390.

Draw *AM*, *CN* ⊥s to *BD*.

Then $\because \angle ADM = \angle NBC$,

and $\angle AMD = \angle BNC$, and $AD = BC$,

$\therefore AM = CN$.

Since then the altitudes of △s *APD*, *CPD* on the same base are equal,

$\therefore \triangle APD = \triangle CPD$;

and for the same reason $\triangle APB = \triangle CPB$.

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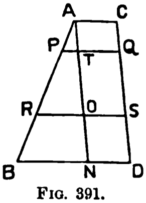

FIG. 391.

EXERCISE 1. Let *AB*, *CD* be any two straight lines cut by the parallels *PQ*, *RS*, *BD*.

Draw *AN* ∥ to *CD*, meeting *BD* in *N*.

Then $BR : RA = NO : OA$, (vi. 2.)

and $\therefore BR : NO = RA : OA$; (v. 15.)

and, similarly, $RP : OT = RA : OA$;

$\therefore BR : NO = RP : OT$ (v. 5.)

$\therefore BR : RP = NO : OT$ (v. 15.)

$= DS : SQ$. (I. 34.)

Ex. 2. Let AB be $\parallel$ to CD.

Draw $AO \parallel$ to BD, cutting PQ in N.

Then $CP : PA = ON : NA$,

$= DQ : QB$.

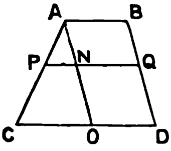

FIG. 392.

Ex. 3. Let AB, CD, EF, GH be four parallel straight lines.

Then $AE : EO = BF : FO$;

$\therefore AE : BF = EO : FO$,

$= CO : DO$,

$= CG : DH$.

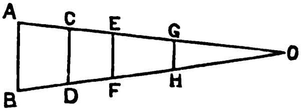

FIG. 393.

Ex. 4. Let $ABCD$ be the quadrilateral; M, N, O, P the middle pts. of the sides. Join AC.

Then $\because AM : MB = CN : NB$;

$\therefore AC$ is $\parallel$ to MN;

and $\because AP : PD = CO : OD$;

$\therefore AC$ is $\parallel$ to PO.

Hence PO is $\parallel$ to MN, and similarly, if BD be joined, we can show that PM is $\parallel$ to ON;

$\therefore MPON$ is a ▱.

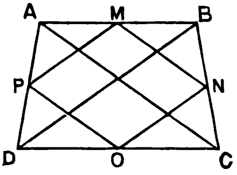

FIG. 394.

Ex. 5. Let $ABCD$ be a trapezium, having $AD \parallel$ to BC.

Produce BA, CD to meet in E.

Draw EN to the intersection of the diagonals, and let it meet the $\parallel$ sides in O, M.

Then $\because \triangle BAC = \triangle CDB$, $\therefore \triangle BNA = \triangle DNC$;

also $\triangle BAN : \triangle BNE = BA : BE$,

$= CD : CE$,

$= \triangle DNC : \triangle CNE$;

$\therefore \triangle BNE = \triangle CNE$.

Again, $\triangle ECN : \triangle MCN = EN : NM$,

$= \triangle BNE : \triangle BNM$;

$\therefore \triangle BNM = \triangle MCN$, and $\therefore BM = CM$.

Now $BM : ME = AO : OE$;

and $CM : ME = DO : OE$;

$\therefore AO = DO$.

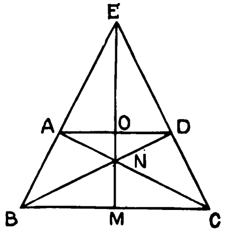

FIG. 395.

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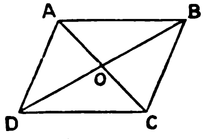

FIG. 396.

EXERCISE 1. Let $ABCD$ be a ▱, and let O be the pt. where the diagonals bisect each other.

Then if OB bisects ABC,

$$AB : BC = AO : OC;$$

and $\therefore AB = BC$; but not otherwise.

Ex. 2. Let BC be the given straight line.

On BC describe the equilateral $\triangle\ BAC$.

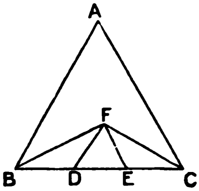

FIG. 397.

Bisect $\angle\ ABC$ and $\angle\ ACB$ by st. lines, meeting in F.

Draw FD, $FE \parallel$ to AB, AC respectively.

Then $\because \angle\ DFB = \angle\ FBA = \angle\ FBD$;

$\therefore FD = BD$, and similarly $FE = EC$.

Now $\triangle\ FDE$ is equiangular to $\triangle\ ABC$, and is therefore an equilateral $\triangle$.

$$\therefore DE = FD = FE;$$

$$\therefore BD = DE = CE.$$

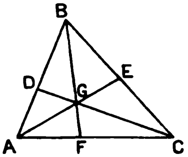

FIG. 398.

Ex. 3. Let EA, DC be bisectors of $\angle$s BAC, BCA, and let them intersect in G.

Join BG, and produce it to meet AC in F.

Then shall BF bisect $\angle\ ABC$.

For $BC : CF = BG : GF$,

$$= BA : AF;$$

$$\therefore BC : BA = CF : AF;$$

and $\therefore FB$ bisects $\angle\ ABC$.

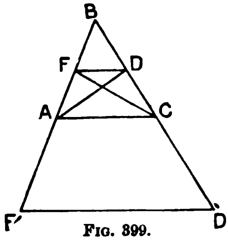

FIG. 399.

Ex. 4. Since $BF : FA = BC : CA$.

$$\therefore BF : BF + FA = BC : BC + CA;$$

$$\therefore BF : BA = BC : BD'.$$

Similarly, $BD : BC = BA : BF'$.

$$\therefore \text{rect. } BF, BD' = \text{rect. } BA, BC,$$

$$= \text{rect. } BD, BF';$$

$$\therefore BF : BD = BF' : BD';$$

$$\therefore FD \text{ is } \parallel \text{ to } F'D'.$$

Ex. 5. Let ABC be an isosceles △. Draw CD, BE, bisecting the ∠s at the base. Join DE.

Then $AD : DB = AC : BC$,

$= AB : BC$,

$= AE : EC$;

∴ DE is ∥ to BC.

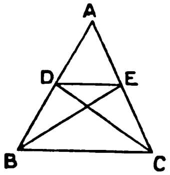

Fig. 400.

Page 250.

Exercise 1. If the angles at the base are equal, the bisector of the exterior angle at the vertex will be parallel to the base. (See p. 52, Ex. 2, and p. 345, Rider set in 1860.)

Ex. 2. Draw BOP ⊥ to CD, and make $OP = OB$. Join PA, cutting CD in D. Join DB.

Then △s DOP, DOB are equal in all respects;

∴ CD bisects ∠ PDB;

∴ $AD : DB = AC : CB$.

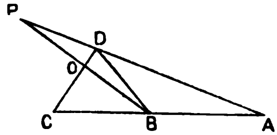

Fig. 401.

Ex. 3. Divide BC in O, so that

$BO : OC = BD : DC$.

Then ∵ $BO : OC = BA : AC$;

∴ AO bisects ∠ BAC;

∴ ∠ OAC + ∠ CAD

$= \frac{1}{2}(\angle BAC + \angle CAE)$,

= a right angle.

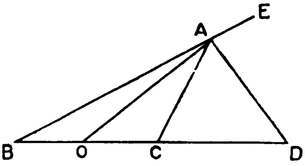

Fig. 402.

Ex. 4. Let AF, AD be the internal and external bisectors; AP the perpendicular to AB.

Then, by Ex. 3, ∠ FAD is a rt. angle;

∴ a circle described on FD as diameter will pass through A;

and since ∠ BAF = ∠ ADF, BA is a tangent to this ⊙;

and AP being ⊥ to AB will be a radius of the ⊙; and ∴ $FP = DP$.

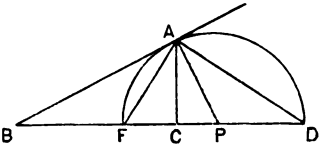

Fig. 403.

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Let BAC be a given $\angle$, and take $AB : AC$ in the given ratio.

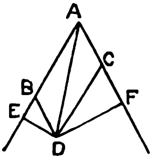

FIG. 404.

Draw $BD \parallel$ to AC, and $CD \parallel$ to AB, and join AD.
Then AD divides the $\angle BAC$ as is required.
Draw DE, $DF \perp$ to AB, AC, or these produced.
Then $\triangle$s BED, CFD are similar;

$$\text{and } \therefore FD : ED = DC : DB,$$
$$= AB : AC;$$

and the perpendiculars drawn from any other pt. in AD will be in the same ratio.

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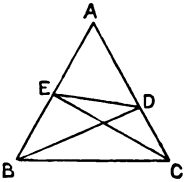

FIG. 405.

EXERCISE 1. Since $\triangle$s ABD, ACE are equiangular,

$$\therefore BA : AD = CA : AE;$$
$$\therefore BA : AC = AD : AE;$$

$\therefore$ $\triangle$s AED, ABC are equiangular, by the Proposition.

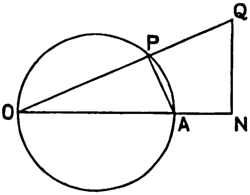

FIG. 406.

Ex. 2. Let OA be the diameter through O.
Draw $QN \perp$ to OA.
Then the $\triangle$s OPA and ONQ are similar;

$$\therefore OP : OA = ON : OQ;$$

$\therefore$ rect. OA, ON = rect. OP, OQ, a constant;

$$\therefore N \text{ is a fixed point};$$

and the locus of Q is a straight line through N perpendicular to ON.

Miscellaneous Exercises on Propositions I. to VI.

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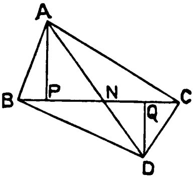

FIG. 407.

1. Let ABC, DBC be the $\triangle$s.
Join AD, and draw AP, $DQ \perp$ to BC.
Let AD cut BC in N.
Then ANP, DNQ are similar $\triangle$s;

$$\therefore \triangle ABC : \triangle DBC = AP : DQ,$$
$$= AN : DN.$$

2. On DC the radius of the $\odot$ ABC, of which AC is a diameter, describe the circle DEC.

Draw BEC cutting the $\odot$, whose diameter is DC, in E.

Then since $\angle ABC =$ a rt. $\angle = \angle DEC$;

$\therefore$ $\triangle$s ABC, DEC are similar;

$$\therefore BE : EC = AD : DC,$$

$$\therefore BE = EC.$$

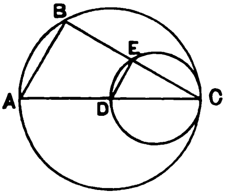

FIG. 408.

3. Since $\triangle$s AFG, CFD are similar,

$$\begin{aligned} DF : FG &= DC : AG, \\ &= BD : AG, \\ &= ED : GE; \end{aligned}$$

$$\therefore DE : DF = GE : FG.$$

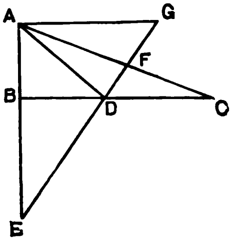

FIG. 409.

4. Since $BA : AC = BD : DC$;

$$\begin{aligned} \therefore BA - AC : BA + AC &= BD - DC : BD + DC, \\ &= 2OD : 2OB, \\ &= OD : OB. \end{aligned}$$

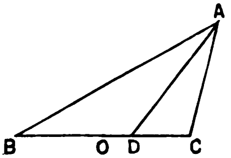

FIG. 410.

5. Let ABC be a triangle, AD the bisector of $\angle BAC$, BD, CE $\perp$s on AD from B and C.

Then $\because$ $\triangle$s ABD, ACE are similar,

$$\therefore BD : AB = CE : AC;$$

and $\therefore$ $BD : CE = AB : AC$.

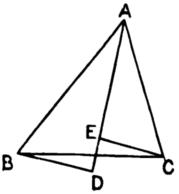

FIG. 411.

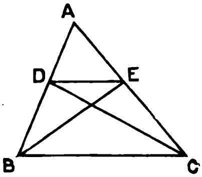

FIG. 412.

6. Take $\triangle ADE$ from each of the equal $\triangle$s DAC, EAB.

Then $\triangle DBE = \triangle ECD$;

$\therefore DE$ is $\parallel$ to BC;

$\therefore BD : DA = CE : EA$.

7. O is the centre of the escribed $\odot$, touching the side BC. (See IV. 4, Ex. 4.)

$\therefore AO$ will bisect $\angle BAC$;

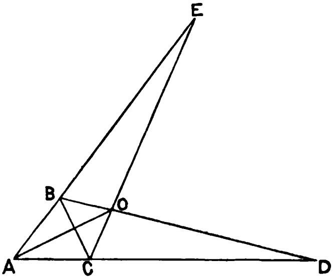

FIG. 413.

$\therefore AD : AB = DO : OB$;

and $OC : OE = AC : AE$.

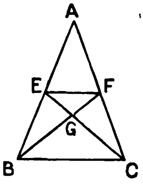

FIG. 414.

8. Since E and F are the middle pts. of AB, AC, EF is $\parallel$ to BC.

Now $\triangle AEC = \triangle BEC$;

and $\triangle EBC = \triangle FCB$; and taking $\triangle BGC$ from each,

$\triangle EBG = \triangle FCG$.

Hence $\triangle AEC - \triangle FCG = \triangle BEC - \angle EBG$;

$\therefore$ fig. $AEGF = \triangle BCG$.

9. Through *O* any point in *BD*, a diagonal of the ▱*ABCD*, draw *FOE*, meeting *AB*, *CD* in *F*, *E*.

Then △s *BOF*, *DOE* are similar.

$\therefore BO : OD = FO : OE.$

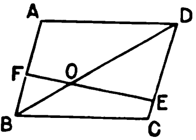

Fig. 415.

10. Since $DF : FE = BD : CE,$

$= DA : AE;$

$\therefore AF$ bisects $\angle BAC$;

$\therefore$ the locus of *F* is a straight line bisecting $\angle BAC$.

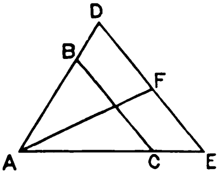

Fig. 416.

Page 258.

Exercise 1. Let *AC* be the given line. Draw *AB* making any angle with *AC*. Make *AD* = seven times *AB*.

Join *CD*, and draw *BE* ∥ to *CD*.

Then $AE : AC = AB : AD$;

$= 1 : 7.$

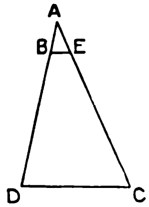

Fig. 417.

Ex. 2. Let *AC* be the given line.

Draw *AB* making any angle with *AC*.

Make *AD* = five times *AB*, and in *BD* take *BE* = *AB*. Join *CD*, and draw *EF* ∥ to *CD*.

Then $AF : AC = AE : AD$;

$= 2 : 5.$

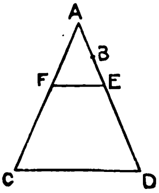

Fig. 418.

Page 259.

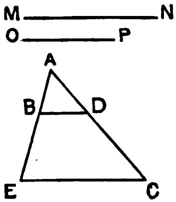

FIG. 419.

EXERCISE 1. Let MN, OP be two lines in the given ratio.

Let AB be the given st. line.

Draw $AC = MN$, making any $\angle$ with AB.

In AC take $CD = OP$.

Join BD and draw CE, meeting AB produced in E, $\parallel$ to BD.

Then $AE : EB = AC : CD$,
$= MN : OP$.

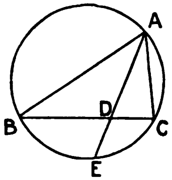

FIG. 420.

Ex. 2. Let BC be the given base. Describe on BC a segment of a $\odot$ BAC, capable of containing the given vertical $\angle$. Bisect the remaining segment BEC in E. Divide BC in D in the given ratio of the sides. Join ED, and produce it to meet the Oce in A. Then BAC is the triangle required.

For since arc BE = arc CE, $\therefore \angle BAD = \angle CAD$.
$\therefore BA : AC = BD : DC$.

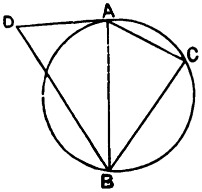

FIG. 421.

Page 261.

EXERCISE. Since $\angle DAB = \angle ACB$, in alternate segment, and $\angle ABD = \angle ABC$,
$\therefore$ $\triangle$s BAD, ABC are similar;
and $\therefore BD : DA = AB : AC$.

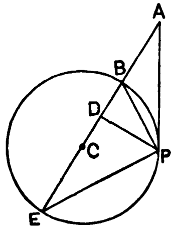

FIG. 422.

Page 262.

EXERCISE. Produce ABC to meet the Oc
E, and join EP.

Then $\angle APB = \angle BEP$ in alternate segn
and since $\triangle BPE$ is right-angled at P,
$\therefore \angle BPD = \angle BEP$; (V
$\therefore \angle APB = \angle DPB$.

Page 263.

Exercise 1. Let AB be the given st. line.

On AB as diameter describe a $\odot$.

Draw $BC \perp$ to AB and $= AB$.

Draw $CDOE$ through O the centre.

Produce AB to F, so that $FB = CD$.

Then rect. AF, $FB =$ rect. EC, CD,
$= $ sq. on BC,
$= $ sq. on AB;

$\therefore$ AB is a mean proportional between AF and FB.

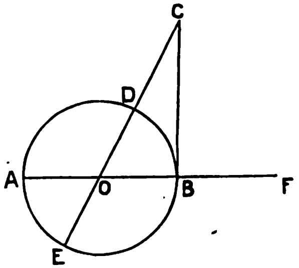

Fig. 423.

Ex. 2. Let ABC be an isosceles $\triangle$.

Draw $AD \perp$ to AB, meeting BC (or BC produced) in D. Draw AE to E the middle pt. of BD.

Then $\because$ BAD is a rt. $\angle$, $\therefore$ E is centre of $\odot$ described about $\triangle ABD$.

$$\therefore \angle EAB = \angle ABE = \angle ACB;$$

$$\therefore \angle AEB = \angle BAC;$$

$$\text{and } \therefore CB : AB = AB : BE.$$

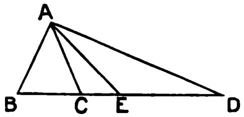

Fig. 424.

Ex. 3. Let DEF be an equilateral $\triangle$ described about the $\odot$ ABC. Through K the centre draw DGB, which bisects EF at rt. $\angle$s. Draw the tangent $LGM \parallel$ to EF; then LM is the side of a regular hexagon described about the $\odot$. Draw the radius KA meeting DE in A.

Now $\triangle$s DAK, DGL are similar;

$$\therefore DA : AK = DG : GL;$$

$$\therefore 2DA : 2AK = 2DG : 2GL.$$

$\therefore$ observing that $DG = KG$, which can easily be shown by joining KL and KM, and proving that $DLKM$ is a rhombus, whose diagonals bisect each other,

$$DE : GB = GB : LM.$$

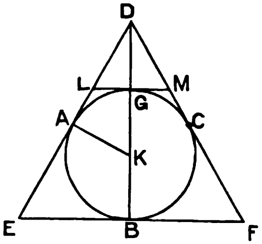

Fig. 425.

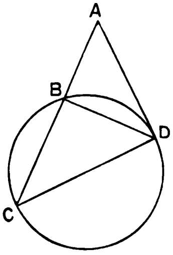

FIG. 426.

Ex. 4. Draw AD a tangent to the ⊙, and join BD, DC.

Then $\because \angle ADB = \angle ACD$, in alternate segment,

$\therefore \triangle$s ADB, ACD are similar;

$\therefore AB : AD = AD : AC.$

Page 266.

EXERCISE 1. Since $CA : AD = EA : AB$,

(1) $CA : EA = AD : AB$; (V. 15.)

(2) $AD : CA = AB : EA$; (V. 12.)

(3) $AD : AB = CA : EA$. (V. 15.)

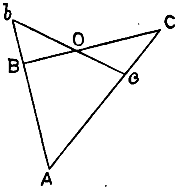

FIG. 427.

Ex. 2. Let O be the intersection of BC, bc.

Then $\because \triangle ABC = \triangle Abc$,

$\therefore \triangle bOB = \triangle cOC$;

$\therefore BO : OC = Oc : Ob.$

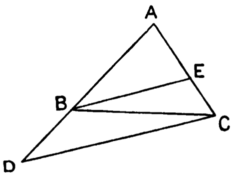

FIG. 428.

Ex. 3. Take AE a mean proportional between AC, CE.

Join EB, and draw $CD \parallel$ to EB.

Then $BE : DC = AE : AC$,

and $BD : AB = EC : AE$.

But, by hypothesis, $AE : AC = EC : AE$;

$\therefore BE : DC = BD : AB$;

and $\angle ABE = \angle BDC$;

$\therefore \triangle ABE = \triangle BCD.$

Ex. 4. Since $AB : AQ =$ duplicate ratio of $AB : AN$;
and $AC : AP =$ duplicate ratio of $AC : AM$,
or, $AP : AC =$ duplicate ratio of $AM : AC$;
and since $AB : AQ = AP : AC$,

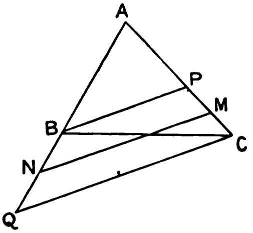

FIG. 429.

$\therefore$ duplicate ratio of $AB : AN =$ duplicate ratio of $AM : AC$;
$\therefore AB : AN = AM : AC$;
and $\angle$ at A is common to the $\triangle$s ANM, ABC;
$\therefore \triangle ANM = \triangle ABC$.

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EXERCISE 1. Draw $CP \perp$ to AB, and $EOF \perp$ to AB.
Then $\triangle$s PCD, CEF are similar;
$\therefore CP : DC = CE : EF$;
$\therefore$ rect. $CP, EF =$ rect. DC, CE;
$\therefore$ rect. $CP, AB =$ rect. DC, CE. (1.)
Again, $EF : CE = DE : OE$;
$\therefore$ rect. $EF, OE =$ rect. CE, DE;
$\therefore$ rect. $OE, AB =$ rect. CE, DE. (2.)
Adding (1) and (2)
rect. $OE, AB +$ rect. $CP, AB =$ sq. on CE;
$\therefore$ 2 area of $\triangle AEB$ + 2 area of $\triangle ACB =$ sq. on CE;
$\therefore$ 2 area of quadrilateral $AEBC =$ sq. on CE.

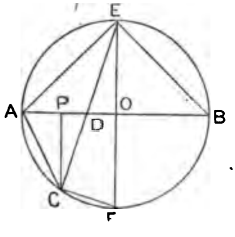


FIG. 430.

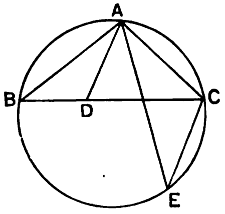


FIG. 431.

Ex. 2. Let ABC be the $\triangle$, AD and AE the lines drawn to meet BC and the $\odot$ described about ABC.

Join EC.

Then $\angle ACE = \angle ADB$, by hypothesis,
and $\angle ABD = \angle AEC$, in same segment,
$\therefore$ $\triangle$s ABD, ACE are similar;
$\therefore AB : AD = AE : AC$;
$\therefore$ rect. AB, AC = rect. AD, AE.

Page 274.

Miscellaneous Exercises.

EXERCISE 1. Let ABC, DEF be the given $\triangle$s.
Produce EF to Q, making $EQ = BC$.

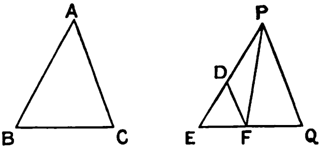


FIG. 432.

Draw $QP \parallel$ to FD, meeting ED produced in P.

Join PF.

Then it may be shown, as in Proposition XIX., that $\triangle EPQ : \triangle EDF$ = duplicate ratio of $EQ : EF$:
and, since $\triangle ABC = \triangle EPQ$,
$\therefore \triangle ABC : \triangle DEF$ = duplicate ratio of $BC : EF$.

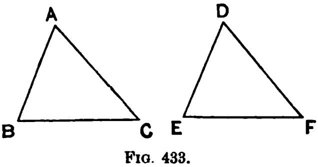


FIG. 433.

Ex. 2. $\triangle ABC : \triangle DEF$ = duplicate ratio of $AC : DF$.

Now $AC : CB = DF : FE$;
$\therefore AC : DF = DF : FE$;
$\therefore AC : EF$ = duplicate ratio of $AC : DF$;
$\therefore \triangle ABC : \triangle DEF = AC : EF$.

Ex. 3. Let ABC be the given $\triangle$.
Construct a square EM = sq. on AB.
Cut off the rectangle $EF = \frac{1}{3}$ of this square.
Describe a square G = rectangle EF.

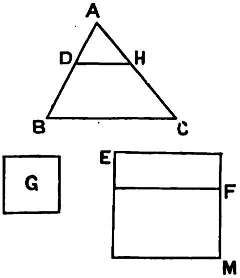


Fig. 434.

Take in AB a pt. D, so that AD = a side of the square last described.
Draw $DH \parallel$ to BC.
Then $\triangle ADH : \triangle ABC$ = sq. on AD : sq. on AB;
= 1 : 3.

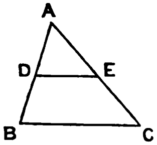


Fig. 435.

Ex. 4. Since $DE = \frac{1}{2} BC$,
$\therefore \triangle ADE = \frac{1}{4} \triangle ABC$;
$\therefore$ quadrilateral $DBCE = 3 \triangle ADE$.

Ex. 5. Let O be the centre of the $\odot$ ABC, and let ABC be an equilateral $\triangle$. Draw $OD \perp$ to AO; draw the diameter BOP, and join AD, AP; these are the sides of the square and the hexagon.

Then sq. on AD = 2 sq. on AO = 2 sq. on AP;
and sq. on BP = sq. on AB + sq. on AP,
or 4 sq. on AP = sq. on AB + sq. on AP;
$\therefore$ sq. on AB = 3 sq. on AP.
$\therefore$ sq. on AP : sq. on AD : sq. on AB = 1 : 2 : 3.

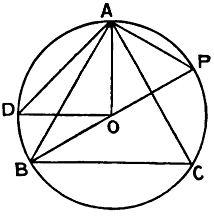


Fig. 436.

Ex. 6. $\triangle AGF : \triangle AHE = FG : HE$,
$= \text{rect. } FG, HE : \text{sq. on } HE$;
and $\triangle AHE : \triangle ADB = \text{sq. on } HE : \text{sq. on } BD$;
$\therefore \triangle AGF : \triangle ADB = \text{rect. } FG, HE : \text{sq. on } BD$; (V. 21.)
$\therefore \triangle AGF : \triangle BDC = \text{rect. } FG, HE : \text{sq. on } BD$. (1.)

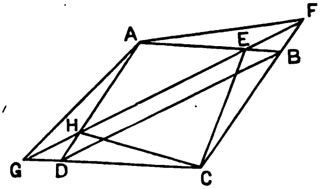


Fig. 437.

Again, $\triangle CFG : \triangle HEC = FG : HE$,
$= \text{sq. on } FG : \text{rect. } FG, HE$;
and $\triangle BDC : \triangle CFG = \text{sq. on } BD : \text{sq. on } FG$;
$\therefore \triangle BDC : \triangle HEC = \text{sq. on } BD : \text{rect. } FG, HE$; (2.)
$\therefore$ from (1.) and (2.) $\triangle HEC = \triangle AGF$.

Ex. 7. Let ABC, DEF be two $\triangle$s having $\angle ABC = \angle DEF$.

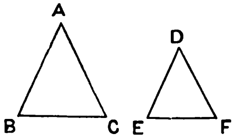


Fig. 438.

Then $\because$
$\triangle ABC : \triangle DEF =$ duplicate ratio of $AB : DE$,
and
$\triangle ABC : \triangle DEF =$ duplicate ratio of $BC : EF$,
$\therefore AB : DE = BC : EF$;
$\therefore \triangle ABC$ is similar to $\triangle DEF$. (VI. 6.)

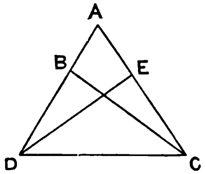


Fig. 439.

Ex. 8. Let the $\triangle$s ABC, ADE have the angle at A common.

Join DC.

Then $\triangle ABC : \triangle ADC = AB : AD$,
$= AB \,.\, AC : AD \,.\, AC$;
and $\triangle ADC : \triangle ADE = AC : AE$,
$= AD \,.\, AC : AD \,.\, AE$;
$\therefore \triangle ABC : \triangle ADE = AB \,.\, AC : AD \,.\, AE$.

Ex. 9. Produce AC to meet BZ produced in D.

Let O be the centre of the ⊙.

Then ∵ AY, OC, BZ are three parallel lines,

∴ $CY = CZ$, since $OA = OB$; (VI. 2, Ex. 2.)

and, since CO is ∥ to BD, and $AO = OB$,

$$\therefore AC = DC.$$

Hence △ACY = △DCZ, (I. 4.)

and △ABC = △DBC; (I. 4.)

∴ △ABC = △BCZ + △DCZ,

= △BCZ + △ACY.

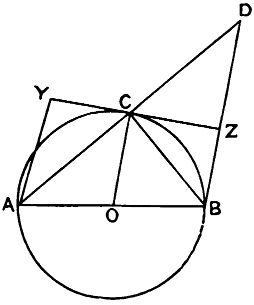


Fig. 440.

Ex. 10. Since ∠ BCD = ∠ CAD, the △s BCD, CAD are similar;

$$\therefore AD : CD = CD : DB;$$

∴ $AD : DB$ = duplicate ratio of $CD : DB$,

= duplicate ratio of $AC : CB$.

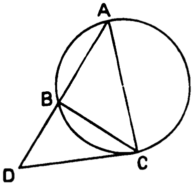


Fig. 441.

Ex. 11. Let $m : n$ be the given ratio.

Let ABC be the given △.

Take any point E in AC. Join BE, and make $BE : ED = n : m$.

Join AD, DC. Then DBC is the △ required.

For △AEB : △BEC = $AE : EC$,

and △AED : △CED = $AE : EC$;

∴ △AEB : △BEC = △AED : △CED;

∴ △ABC : △BEC = △ADC : △CED;

∴ △ABC : △ADC = △BEC : △CED,

= $BE : ED$,

= $n : m$.

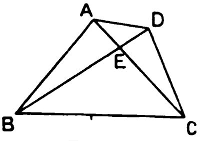


Fig. 442.

Page 285.

Exercise 1. Area of △ABC = $\frac{1}{2}$ rect. AD, BC.

Now BC can never be greater than the diameter EA;

∴ rect. EA, AD cannot be less than rect. AD, BC;

∴ rect. BA, AC cannot be less than rect. AD, BC;

∴ rect. BA, AC cannot be less than twice area of △ABC.

Ex. 2. The line joining OO' bisects AP the common chord of the $\odot$s at rt. $\angle$s. (See. p. 24, Ex. 8.)

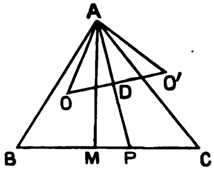


FIG. 443.

Also $\angle AOD$, being half the $\angle AOP$, is equal to $\angle ABP$;

$\therefore$ $\triangle$s AOD, ABM are similar;

and $\therefore$ rect. OD, AM = rect. AD, BM.

Similarly, rect. $O'D$, AM = rect. AD, MC.

$\therefore$ rect. OO', AM = rect. AD, BC,

$= \frac{1}{2}$ rect. AP, BC.

Ex. 3. This is precisely the same as Prop. B.

Page 286.

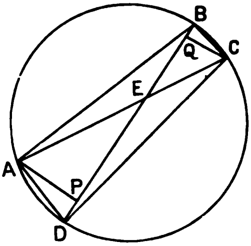


FIG. 444.

EXERCISE. Let $ABCD$ be a quadrilateral inscribed in a $\odot$, and let the diagonals intersect at E, so that

$$\angle AED = \angle BEC = \tfrac{1}{3} \text{ of a rt. } \angle.$$

Draw AP, CQ $\perp$s to BD.

Then $AE = 2AP$ and $EC = 2CQ$.

(P. 116, Ex. 3.)

Then area $\triangle ABD = \frac{1}{2}$ rect. AP, $BD = \frac{1}{4}$ rect. AE, BD,

and area $\triangle BCD = \frac{1}{2}$ rect. CQ, $BD = \frac{1}{4}$ rect. EC, DB;

$\therefore$ area of $ABCD = \frac{1}{4}$ rect. AC, BD;

$\therefore$ 4 area of $ABCD$ = rect. AB, CD + rect. AD, BC.

Page 288.

EXERCISE. Taking the diagram of the Proposition

$FC : GK = CK : KD$,

$= CK : GF$,

$= KF : AG$ (by similar $\triangle$s FKC, AGF),

$= DG : AG$,

$= GK : AF$;

and similarly for the other complement.

Page 291.

EXERCISE 1. Let $ABCDE$ be a regular pentagon.

Join AC, BD meeting in P.

Then since AC is $\parallel$ to ED (IV. 11, Ex.), and BD is $\parallel$ to AE,

$\therefore APDE$ is a rhombus.

Now $\triangle ABC$ is similar to $\triangle BPC$;

$\therefore AC : CB = BC : CP$;

and $CB = AE = AP$;

$\therefore AC : AP = AP : CP$.

Similarly, $BD : PD = PD : BP$.

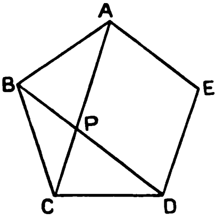


FIG. 445.

Ex. 2. Take the diagram of IV. 10.

Then $BD = AC$, and we have to show that BD is the side of a regular decagon inscribed in $\odot\ BDE$.

Now this will be the case if $\angle BAD = \frac{1}{10}$ of four rt. $\angle$s;

and since $\angle BAD = \frac{1}{5}$ of two rt. $\angle$ s,

$\angle BAD$ is $= \frac{1}{10}$ of four rt. $\angle$s.

Miscellaneous Exercises on Book VI.

Page 293.

1. Let O, O' be the centres of the circles, APQ one of the common tangents.

Then $AO : AO' = OP : O'Q$,

$= 1 : 3$;

$\therefore OO' = 2OA$;

$\therefore OA =$ diameter of smaller circle;

and $AO' = \frac{3}{2}\ OO' =$ diameter of larger circle.

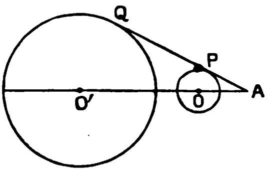


FIG. 446.

2. Draw FE, ED, DF through A, B, C, the angular points of $\triangle ABC$, and $\parallel$ to BC, CA, AB.

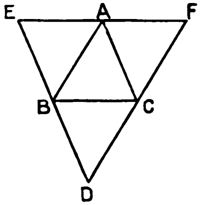


FIG. 447.

Then $\because ABCD$ is a ▱,

$\therefore \angle BDC = \angle BAC$.

Similarly, $\angle AEB = \angle ACB$,

and $\angle AFC = \angle ABC$.

Hence $\triangle DEF$ is similar to $\triangle ABC$,

and $\because AF = BC$, and $AE = BC$,

$\therefore AE = AF$;

and similarly, $EB = BD$, and $FC = CD$.

So also, if lines be drawn through D, E, F $\parallel$ to EF, FD, DE, a $\triangle$ will be formed similar to EDF, and having its sides bisected in D, E, F.

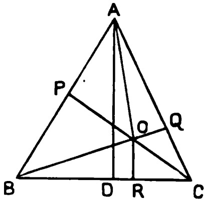


FIG. 448.

3. Let ABC be an equilateral $\triangle$, O any pt. in it. Draw AD, OP, OQ, OR $\perp$s to the opposite sides.

Then $\triangle ABC : \triangle AOB = AD : OP$;

$\triangle ABC : \triangle AOC = AD : OQ$;

$\triangle ABC : \triangle BOC = AD : OR$;

$\therefore \triangle ABC : \triangle AOB + \triangle AOC + \triangle BOC$

$= AD : OP + OQ + OR$;

$\therefore AD = OP + OQ + OR$.

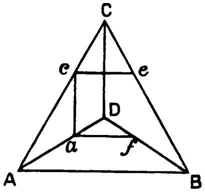


FIG. 449.

4. Let ca be the line $\parallel$ to CD, and let ce, af be drawn $\parallel$ to AB.

Then $af : AB = Da : AD$,

$= Cc : CA$,

$= ce : AB$;

$\therefore af = ce$.

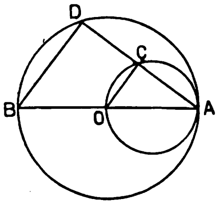


FIG. 450.

5. Let DCA be any chord drawn through A.

Then $\angle$s at C and D are right $\angle$s;

$\therefore \triangle$s ACO, ADB are similar;

$\therefore AC : CD = AO : OB$;

$\therefore AC = CD$.

6. Let BC be the given base. Describe on BC the segment of a ⊙ BAC capable of containing the given vertical ∠. Bisect the remaining segment in E. Take D in BC such that $BD=2DC$. Join ED and produce it to A.

Then $\angle BAD = \angle CAD$, and

$\therefore BA : AC = BD : DC = 2 : 1.$

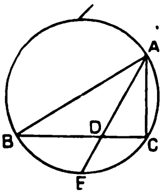


Fig. 451.

7. Since EF is $\parallel$ to AC,

$\therefore AE : EB = CF : FB$;

and $\because EG$ is $\parallel$ to AD,

$\therefore AE : EB = DG : GB$;

$\therefore CF : FB = DG : GB$;

and $\therefore FG$ is $\parallel$ to CD.

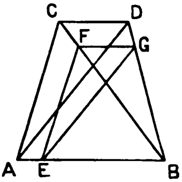


Fig. 452.

8. Let O, P be the pts. in which AD, CF cut BC, AB.

Join AF, DC.

Then $\because$ ∠s at P and O are right ∠s,

$\therefore \angle LCO = \angle PAL$,

$= \angle OCD$, in same segment;

$\therefore$ △s LOC, DOC are equal in all respects;

$\therefore LO = OD$;

and similarly for LE, LF.

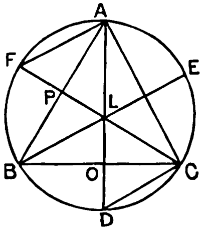


Fig. 453.

9. Since $DE=\frac{1}{3}$ of BC,

$\therefore \triangle ADE=\frac{1}{9}$ of $\triangle ABC$;

and $\triangle DOE=\frac{1}{9}$ of $\triangle BOC$.

Also $\triangle DOB = \triangle EOC$.

Now $\triangle DOB + \triangle DOE + \triangle EOC + \triangle BOC =$ fig. $DBCE$;

$\therefore 2\triangle DOB + 10\triangle DOE = \frac{8}{9}\triangle ABC$;

$\therefore \triangle DOB + 5\triangle DOE = \frac{4}{9}\triangle ABC$.

Again, $\triangle DOB + \triangle DOE = \triangle ABE - \triangle ADE$,
$= \frac{1}{3}\triangle ABC - \frac{1}{9}\triangle ABC$,
$= \frac{2}{9}\triangle ABC$.
$\therefore 2\triangle DOB + 2\triangle DOE = \frac{4}{9}\triangle ABC$. (2.)
Hence from (1.) and (2.),
$\triangle DOB + 5\triangle DOE = 2\triangle DOB + 2\triangle DOE$;
$\therefore 3\triangle DOE = \triangle DOB$;
$\therefore BO = 3OE$.
Similarly, $CO = 3OD$.

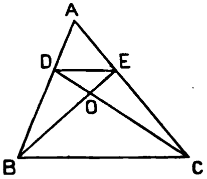


FIG. 454.

10. Since $\triangle FHG = \triangle AFG - \triangle AFC - \triangle CHG$,
and $\triangle BCH = \triangle ABG - \triangle ABC - \triangle CHG$,
$\therefore \triangle FHG - \triangle BCH = \triangle AFG - \triangle ABG - \triangle AFC + \triangle ABC$;
$\therefore \triangle FHG - \triangle BCH : \triangle ADE$
$= \triangle AFG - \triangle ABG - \triangle AFC + \triangle ABC : \triangle ADE$;

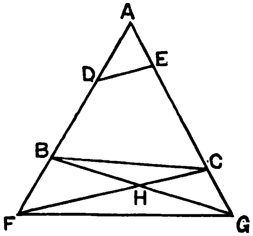


FIG. 455.

$\therefore$, by Ex. 8 on p. 274, $\triangle FHG - \triangle BCH : \triangle ADE$
= rect. AF, AG − rect. AB, AG − rect. AF, AC + rect. AB, AC : rect. AD, AE,
= rect. AD, AG − rect. AD, AC : rect. AD, AE.
$= AG - AC : AE$,
$= CG : AE$,
$= AE : AE$;
$\therefore \triangle FHG - \triangle BCH = \triangle ADE$;
$\therefore \triangle FHG = \triangle BCH + \triangle ADE$.

11. See p. 251, Ex. 4.

12. Let ABC, DEF be equal $\triangle$s on equal bases, and between the same $\|$s AD, BF.

Draw $MNOP \parallel$ to BF.

Then $\triangle AMN : \triangle ABC$
$=$ duplicate ratio of $AN : AC$,
$=$ duplicate ratio of $DP : DF$,
(VI. 2, Ex. 1.)
$= \triangle DPO : \triangle DEF$;
$\therefore \triangle AMN = \triangle DPO$.

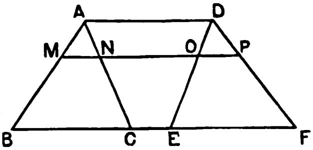


FIG. 456.

13. Bisect AD in T, and draw DER to meet AB in R.

Then since $\angle ACD$ is a rt. $\angle$,
T is the centre of the $\odot$ described about $\triangle ACD$;
$\therefore TC = TA$;
$\therefore TC$ is tangent at C.

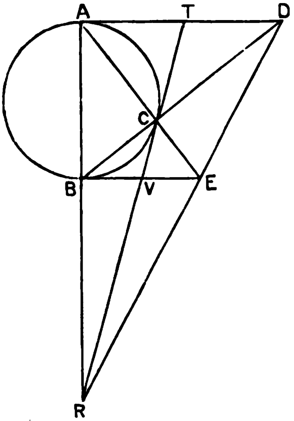


FIG. 457.

Hence CV bisects BE.
$\therefore TV$ produced passes through R.

14. Draw any radius OA of the smaller ⊙.
Produce it to B so that $OB=4OA$.
Describe a semicircle on AB, cutting the larger ⊙ in C.

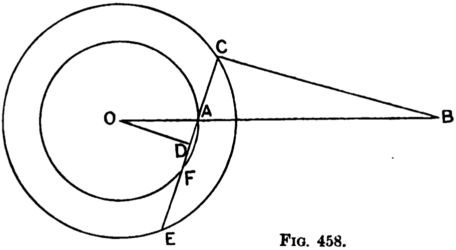


FIG. 458.

Draw $CAFE$ a chord of both circles.
Join CB and draw $OD \perp$ to AF.
Then ∵ △s AOD, ABC are similar,
∴ $CD=4DA$; and ∴ $CE=4AF$.

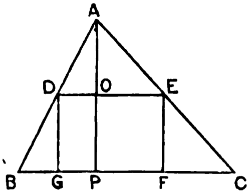


FIG. 459.

15. Draw $AP \perp$ to BC. Bisect BP in G and CP in F.

Then $GF=\frac{1}{2}BC$. Draw GD, $FE \perp$s to BC, to meet AB and AC in D, E. Join DE, cutting AB in O.

Then ∵ $EO=\frac{1}{2}PC$, ∴ $PO=\frac{1}{2}AP$;
∴ area of rect. $DEFG$ = rect. DG, GF,
$=\frac{1}{4}$ rect. AP, BC,
$=\frac{1}{2}$ area of △.

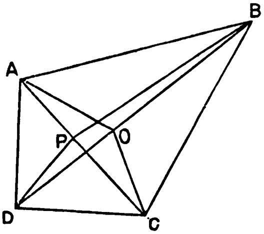


FIG. 460.

16. Let the bisectors of A and C meet in O.

Then $DA : AB=DO : OB$,
$=DC : CB$;
∴ $DA : DC=AB : CB$.

Now bisect ∠ ADC by DP, meeting AC in P, and join PB.

Then $DA : DC=AP : PC$;
∴ $AB : CB=AP : PC$;
∴ PB bisects ∠ ABC.

17. Let $ABCD$ be a quadrilateral described about a $\odot$, and let AD be $\parallel$ to BC, and join E, F the pts. of contact of AB, DC.

Then since $AE : EB = DF : FC$;

$\therefore$ EF is $\parallel$ to AD and BC.

But tangents make equal $\angle$s with the chord of contact;

$\therefore$ AB, DC are equally inclined to EF;

$\therefore$ AB, DC are equally inclined to AD and BC.

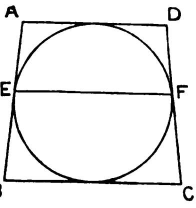


Fig. 461.

18. Let ABC, DBC be $\triangle$s on the same base BC.

Join AD, and produce it to meet BC produced in E.

Draw AM, DN $\perp$s to BE.

Then $\triangle$s AME, DNE are similar;

$\therefore$ $AE : DE = AM : DN$,

$= \text{area} \triangle ABC : \text{area} \triangle DBC$;

the triangles having the same base, and therefore their areas being proportional to their altitudes.

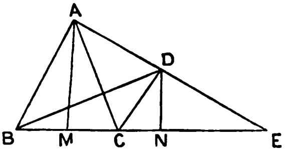


Fig. 462.

19. Let AB, AC be the tangents, and BC the chord of contact.

Draw OD, OE, OF $\perp$s from O, any pt. on the $\bigcirc$ce, to BC, AC, AB. Join ED, FD, BO, CO.

Then $\because$ the $\angle$s at D, E, F are rt. $\angle$s, $DOFB$, $DOEC$ can have $\odot$s described about them.

$\therefore$ $\angle ODE = \angle OCE$, and $\angle OBD = \angle OFD$.

But $\angle OCE = \angle OBD$ in alternate segment;

$\therefore$ $\angle ODE = \angle OFD$, and similarly it may be shown that $\angle OED = \angle ODF$,

$\therefore$ the $\triangle$s ODE, OFD are similar;

$\therefore$ $FO : OD = OD : OE$;

$\therefore$ sq. on OD = rect. FO, OE.

(M'Dowell's *Exercises on Euclid*, p. 98.)

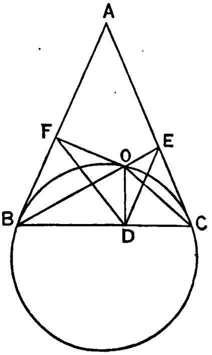


Fig. 463.

20. The two quadrilaterals $EADC$, $CABE$ are similar, for

$$\angle EAD = \angle CAB,$$

and $\angle ADC = \angle ABE$, and also

$$EA : AD = CA : AB,$$

and $AD : DC = AB : BE$;

and they have one side EC common,

$\therefore$ they are equal in all respects; (VI. 21.)

$\therefore AD = AB$, and $AE = AC$,

and $\triangle ABC = \triangle ADE$, and $\triangle ABE = \triangle ACD$.

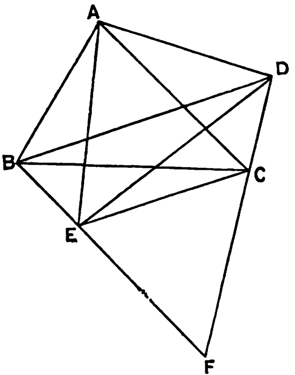


FIG. 464.

Also $BE = CD$, and $\angle ABE = \angle ADC$, and $\angle ABD = \angle ADB$;

$\therefore \angle DBF = \angle BDF$, and $\therefore BF = DF$,

$\therefore EF = FC$, and $\therefore EC$ is $\parallel$ to BD, and the $\triangle$s CFE, BFD are similar.

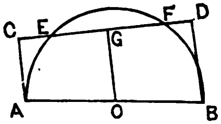


FIG. 465.

21. Let AB be the diameter, and draw AC, BD $\perp$s to the st. line CD, which meets the ○ce in E and F. Then shall $CE = FD$.

From O the centre draw $OG \perp$ to CF.

Then $CG : GD = AO : OB$; (VI. 2, Ex. 1.)

$\therefore CG = GD$;

and $GE = GF$, because OG bisects EF;

$\therefore CE = FD$.

22. Let AB be the given chord.

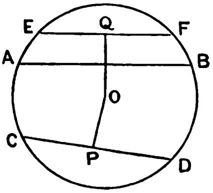


Fig. 466.

Take O the centre of the ⊙, and place in the ⊙ a line CD having the given ratio to AB. Draw $OP \perp$ to CD, and draw OQ cutting AB at rt. ∠s and equal to OP. Through Q draw the chord $EF \parallel$ to AB. Then $EF = CD$, being equally distant from the centre, and therefore EF is in the required ratio to AB, and it is $\parallel$ to AB.

23. Since ∠ EAC = ∠ $ACB = \frac{2}{3}$ of a rt. ∠, and ∠ ACE is a rt. ∠,
∴ ∠ $AEC = \frac{1}{3}$ of a rt. ∠ ;
and ∴ $AE = 2AC$; (See p. 116, Ex. 3.)
and $OF = 2 . OC$,
and $KO = AO = OC$.

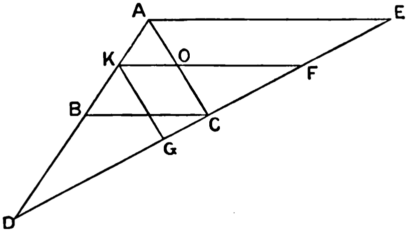


Fig. 467.

Now sq. on KF = sq. on KO + sq. on OF + 2 rect. KO, OF,
= sq. on CO + sq. on OF + 2 rect. CO, OF,
= $\frac{1}{3}$ sq. on CF + $\frac{4}{3}$ sq. on CF + 4 sq. on CO,
= $\frac{1}{3}$ sq. on CF + $\frac{4}{3}$ sq. on CF + $\frac{4}{3}$ sq. on CF,
= 3 sq. on CF,
= 3 sq. on EF ;
∴ sum of sqq. on KG, FG = 3 sq. on EF.

24. Let ABC be the △, having the rt. ∠ at B. Draw $CE \perp$ to AC, meeting AB produced in E.

Make ∠ ECD = ∠ CAB, and let CD meet AE produced in D.

Draw $DF \parallel$ to EC, meeting AC produced in F.

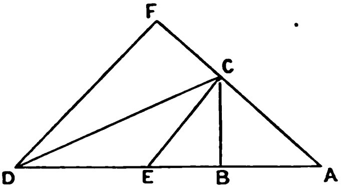


FIG. 468.

Then

$\angle FDC = \angle DCE = \angle CAB.$

$\therefore$ $\triangle$s DCF, ACB are similar,

and $\triangle$s DAF, ACB are similar;

$\therefore FD : DA = BC : CA,$

and $DC : FD = CA : AB$;

$\therefore CD : DA = BC : AB.$

25. Take CD in AD equal to DB, and take BQ in AB produced a third proportional to AC, DB.

With centre Q and distance QD describe a semicircle DRP.

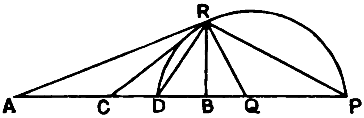


FIG. 469.

Take any pt. R in the $\odot$ce.

Then $AC : DB = DB : BQ$;

$\therefore AD : DB = DQ : BQ$; (V.16.)

$\therefore AD : DQ = DB : BQ$;

$\because AQ : DQ = DQ : BQ$; (V.16.)

$\therefore AQ : QR = QR : BQ.$

Hence ARQ, RQB are similar $\triangle$s; (VI. 6.)

$\therefore AR : RB = AQ : RQ$;

$\therefore AR : RB = AQ : DQ,$

$= DQ : BQ,$

$= AD : DB$;

$\therefore DR$ bisects $\angle ARB$;

and $\therefore$, since RP is $\perp$ to DR,

$AP : PB = AD : DB.$

26. Let G be the point of contact of the circles ABG, DEG, of which AB, DE are parallel diameters, and O, C the centres.

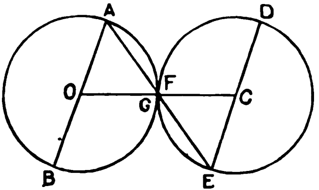


FIG. 470.

Then OC passes through G.

Let OC meet AE in F.

Then $\triangle$s OAF, CEF are similar;

$\therefore AO : CE = OF : FC$;

$\therefore AO + CE : CE = OF + FC : FC$;

$\therefore OC : CE = OC : FC$;

$\therefore FC = CE = CG$;

$\therefore F$ and G are coincident.

27. Let ⊙s ABE, CDE touch each other and also touch the st. line AC.

Draw the diameters AB, CD, and join AD, BC; these lines (as is proved in Ex. 26) pass through E, the point of contact of the ⊙s.

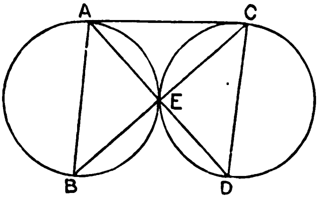


FIG. 471.

Then $\angle CAD = \angle ABE$, in alternate segment;

$\therefore$ △s ABC, CAD are equiangular;

$\therefore BA : AC = AC : CD$.

28. Let AEB, AFC be the circles; O, P their centres.

Draw the common diameter $AOBPC$.

Draw AEF any chord of both circles, and join BE, CF.

Then AEB and AFC are rt. angles;

$\therefore EB$ is $\parallel$ to FC;

$\therefore FA : AE = CA : AB$,

$= 3 : 1$.

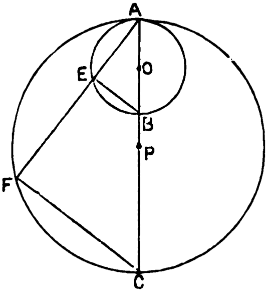


FIG. 472.

29. Make $\angle BCP = \angle BCD$. Then P is the point required.

For, since CB bisects $\angle DCP$,

$PB : BD = CP : CD$;

and, since AC bisects the external $\angle$ of △PCD,

$\therefore PA : AD = CP : CD$.

Hence $PA : AD = PB : BD$;

and $\therefore PA : PB = AD : DB$.

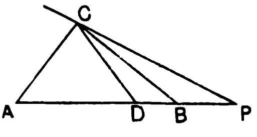


FIG. 473.

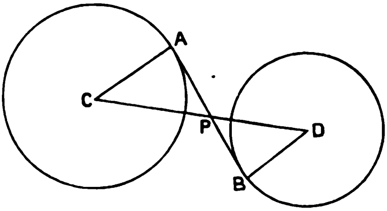


FIG. 474.

30. Let C and D be the centres of the ⊙s, AB a common tangent at A and B.

Let CD and AB intersect in P.

Then, since ∠s at A and B are rt. ∠s, the △s APC, BPD are similar;

$$\therefore CP : PD = AC : BD$$
$$= 2AC : 2BD.$$

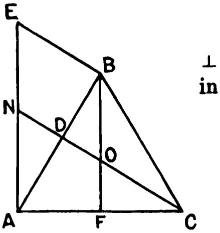


FIG. 475.

31. Let ABC be an equilateral △.

Draw $BE \perp$ to AB, $AE \perp$ to AC, and BF, CD $\perp$s to AC, AB, and let CD produced meet AE in N.

Then $ENOB$ is a ▱, and ∴ $NO = EB$.

Now since OF is ∥ to AN,

$$NO : OC = AF : FC;$$

and ∴, since $AF = FC$,

$NO = OC$; and ∴ $EB = CO =$ radius of circumscribing ⊙.

32. The parallelograms being equal to each other, the rectangles on the same bases and with the same altitudes are equal.

∴ the altitudes are inversely proportional to the bases, that is, to the diagonals of the parallelogram.

33. Since $\triangle FAG : \triangle FGD = AG : GD$,

and $\triangle EAG : \triangle EGD = AG : GD$;

$$\therefore \triangle EAG : \triangle FAG = \triangle EGD : \triangle FGD;$$

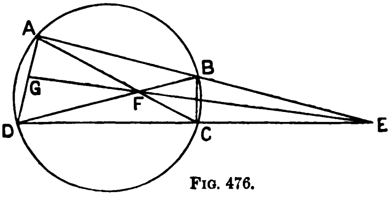


FIG. 476.

$$\therefore \triangle EAF : \triangle FAG = \triangle EFD : \triangle FGD; \qquad \text{(V. 25.)}$$
$$\therefore \triangle EAF : \triangle EFD = \triangle FAG : \triangle FGD;$$
$$\therefore \triangle EAF : \triangle EFD = AG : GD.$$

Hence $AG : GD =$ rect. EA, FA : rect. ED, DF. (See p. 274, Ex. 8.)
But from similar △s AFB, DFC,

$$AB : CD = AF : FD;$$

∴ rect. EA, AB : rect. $ED, DC =$ rect. EA, AF : rect. ED, FD,

$$= AG : GD.$$

34. Let AOB be the diameter of a ⊙, and O the centre.

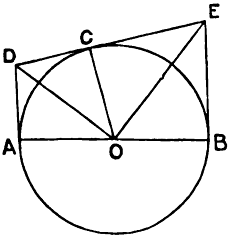


FIG. 477.

Draw AD, BE tangents at A and B, meeting a tangent at any pt. C in D and E.

Then △s COE, BEO are equal.

Since ∠ ECO = ∠ OBE = a rt. angle, and $OB = OC$;

∴ ∠ COE = ∠ EOB.

So also ∠ COD = ∠ DOA.

Hence ∠ DOE is a rt. angle;

and ∵ in the right-angled △ DOE, OC is drawn ⊥ to DE,

$$\therefore DC : CO = CO : CE.$$

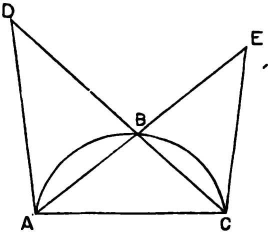


FIG. 478.

35. Let ABC be the △, and draw AD, CE, making ∠ DAC = ∠ ABC,

and ∠ ECA = ∠ ABC.

Then △s ADC, BAC are similar,
and △s AEC, BAC are similar;
∴ △s ADC, AEC are similar;
∴ $AD : AC = AC : CE$;
∴ rect. AD, CE = sq. on AC.

36. Let $ABCD$ be the given parallelogram, EF the given altitude.

Draw EH, FG ⊥s to EF.

Make ∠ GFO = ∠ ADC, O being a pt. in EH.

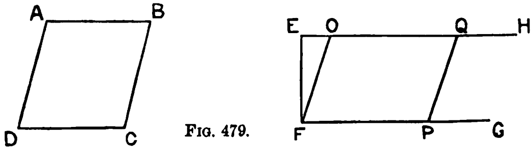


FIG. 479.

Take FP a fourth proportional to OF, AD, DC.

Draw $PQ \parallel$ to OF.

Then ▱ $OFPQ$ = ▱ $ABCD$; (VI. 14.)

∴ $OFPQ$ is the ▱ required.

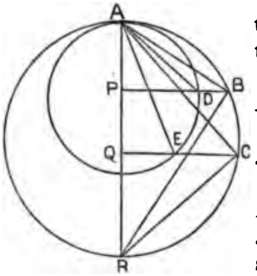


FIG. 480.

37. Let AR be the line drawn from A, the point of contact, through the centres of the ⊙s.

From P and Q draw PDB, QEC ⊥s to AR.

Then $AR : AB = AB : AP$; (VI. 12.)

$\therefore$ $AR : AP$ = duplicate ratio of $AR : AB$.

Similarly, $AQ : AR$ = duplicate ratio of $AC : AR$;

$\therefore$ $AQ : AP$ = duplicate ratio of $AC : AB$;

so also, $AQ : AP$ = duplicate ratio of $AE : AD$;

$\therefore$ $AC : AB = AE : AD$.

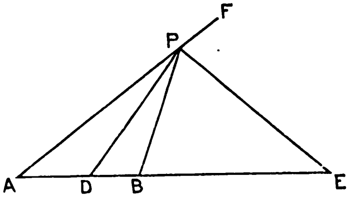


FIG. 481.

38. Let AB be the base, and E the point in which the line bisecting the exterior angle meets the base.

Divide AB in D, so that $AD : DB = AE : BE$; then D is the point in which the line bisecting the interior vertical angle meets the base, and is a fixed point.

Also, if P be the vertex, $\angle DPB = \frac{1}{2} \angle APB$, and $\angle EPB = \frac{1}{2} \angle BPF$;

$\therefore$ $\angle DPE = \frac{1}{2} \angle APB + \frac{1}{2} \angle BPF$ = a rt. angle;

$\therefore$ locus of P is the circle on DE as diameter.

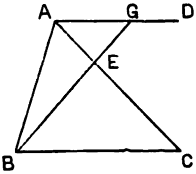


FIG. 482.

39. Let ABC be the given △.

Draw AD ∥ to BC. Divide AC in E so that $AC : CE$ in the given ratio.

Join BE and produce it to meet AD in G.

Then △s AEG, BEC are similar;

$\therefore$ $EG : BE = EA : CE$;

$\therefore$ $BG : BE = AC : CE$; (V. 16.)

$\therefore$ $BG : BE$ = the given ratio.

40. Since $\triangle$s ABD, ACD are similar, and $\angle$ at D is common to both,

$\therefore \angle ACD = \angle BAD$;
and $\angle DAC = \angle ABC$;
Hence $AD : CD = BD : AD$;
$\therefore$ sq. on AD = rect. CD, BD;
$\therefore$ DA touches the ⊙ described about $\triangle ABC$.

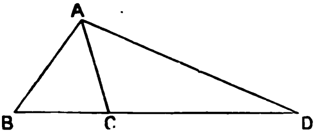


FIG. 483.

41. Let T be any pt. in the line through $P \perp$ to AB,
PR, PQ the equal tangents from P,
TS, TV the tangents from T.

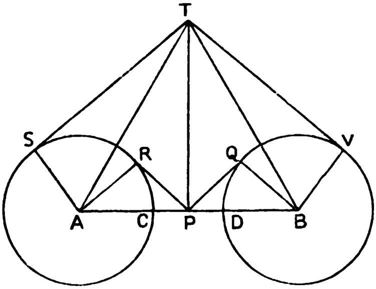


FIG. 484.

Then sqq. on TS, AS, BD = sqq. on TA, BD,
= sqq. on TP, AP, BD,
= sqq. on TP, AR, RP, BD,
= sqq. on TP, AC, PQ, BQ,
= sqq. on TP, AC, BP,
= sqq. on BT, AC,
= sqq. on BV, TV, AC;
$\therefore$ sq. on TS = sq. on TV.

42. Let ABC be the triangle, OP, OQ, OR the three lines. Describe on AB a segment of ⊙ containing ∠ QOP,
on AC „ „ ∠ POR,
on BC „ „ ∠ QOR.

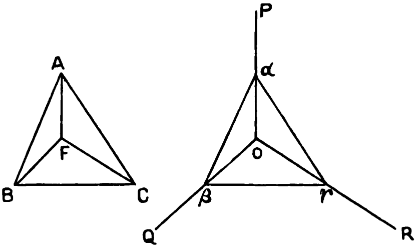


FIG. 485.

These will intersect in one pt. F, since the sum of the three ∠s is four right ∠s. Then take $O\alpha$, $O\beta$, $O\gamma$ equal to AF, BF, CF respectively, and △ $\alpha\beta\gamma$ is not only similar but equal to △ ABC, and any number of similar △s may be formed by starting at any point in OP and drawing straight lines || to $\alpha\gamma$, $\gamma\beta$, $\beta\alpha$.

43. Let BE, DF, the ⊥s on AD, AB, or these produced, meet in O. Join OA, and produce it to meet BD in G.

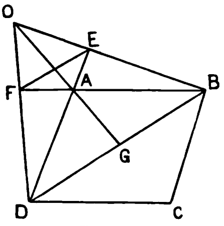


FIG. 486.

Then shall OG be ⊥ to BD.

For, if FE be joined, since a ⊙ can be described about $OFAE$, the angles at E and F being right ∠s,

$$\angle FOG = \angle FED \text{ in the same segment.}$$

And since a ⊙ can be described about $DFEB$,

$$\angle FED = \angle FBD \text{ in the same segment;}$$
$$\therefore \angle FOG = \angle FBD;$$
$$\text{and } \angle OAF = \angle BAG;$$
$$\therefore \angle AGB = \angle OFA = \text{a right } \angle.$$

44. Let $ABCD$ be a quadrilateral inscribed in a ⊙; join AC, BD. Make ∠ ABF = ∠ DBC, then ∠ ABD = ∠ EBC, and ∴ AD=FC. Then since ∠ ADB = ∠ ECB in same segment, the △s ABD, EBC are similar;

∴ rect. BD, BE=rect. AB, BC.

Again, $\angle ECF = \angle ABF = \angle DBC$; and $\angle EFC = \angle CDB$;

$\therefore$ $\triangle$s ECF, BDC are similar;

and $\therefore$ rect. FE, BD = rect. CF, CD = rect. AD, CD.

Hence rect. BF, BD = rect. AB, BC + rect. AD, CD.

Similarly, by taking $\angle BCG = \angle ACD$, we may show that rect. CG, CA = rect. AB, AD + rect. CB, CD.

And since $\angle BCG = \angle ACD$,

$\therefore$ arc BG = arc AD = arc FC;

$\therefore$ arc BAF = arc GFC;

$\therefore$ $BF = GC$;

$\therefore$ rect. CG, CA = rect. BF, CA.

Then $AC : BD$ = rect. AC, BF : rect. BD, BF,

= rect. AB, AD + rect. CB, CD : rect. AB, BC + rect. AD, CD.

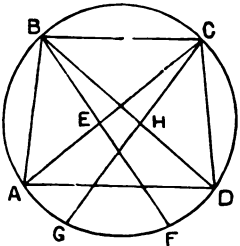


FIG. 487.

45. Let ABC be the $\triangle$, and $BDEC$ the square.

Let AD, AE cut the base in P, Q.

Draw $AF \perp$ to BC. Then $PF = QF$.

Now, by similar $\triangle$s APF, DPB,

$$PF : PB = AF : BD,$$
$$= AF : BC;$$
$$\therefore PF : AF = PB : BC;$$
$$\therefore 2PF : 2AF = PB : BC;$$
$$\therefore PQ : 2AF = PB : BC;$$
$$\therefore PQ : PB = 2AF : BC.$$

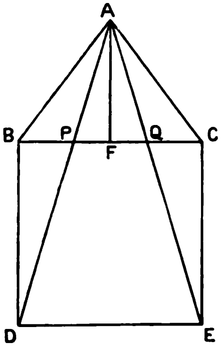


FIG. 488.

46. Since rect. DA, BE = sq. on AC,

$$\therefore DA : AC = AC : BE,$$
$$= CB : BE;$$

and $\angle DAC = \angle CBE$;

$\therefore$ $\triangle$ DAC is similar to $\triangle$ CBE. (VI. 6.)

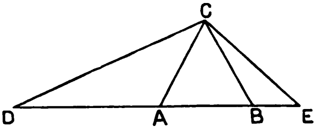


FIG. 489.

47. Bisect the $\angle$s at B and A by st. lines meeting in O, and let BO meet AC in b. Then $\angle\, bBA=\frac{1}{2}\angle\, CBA = \angle\, BAC$; and $Bb=Ab$; and similarly, $Ca=Aa$.

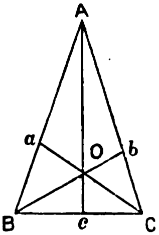


FIG. 490.

Now since AO bisects $\angle\, BAb$,

$\therefore\ BA : BO = Ab : Ob$;

and $\therefore\ BA-BO : BO = BO : Ob$. (1.)

Again, $\because\ CO$ bisects $\angle\, BCb$,

$\therefore\ BC : BO = Cb : Ob$,

$= AB - Bb : Ob$;

$\therefore\ BC+BO : BO = AB-BO : Ob$. (2.)

From (1.) and (2.),

$BC+BO : BA-BO = AB-BO : BO$;

$\therefore\ BC+BA : BA-BO = AB : BO$;

$\therefore\ BC+BA : AB = BA-BO : BO$;

$\therefore\ BC+2AB : AB = AB : BO$;

i.e. perimeter : side = side : distance of cent. of inscr. ⊙ from B.

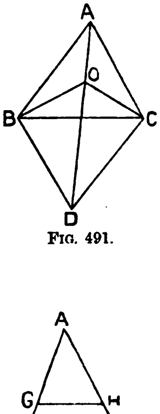


FIG. 491.

48. Let D be the centre of the ⊙ escribed to ABC, which touches BC externally. Then AOD is a straight line, because AD, AO both bisect the angle BAC; also, OB, OC bisect the interior angles at B and C; and BD, CD bisect the exterior angles at B and C; therefore OBD, OCD are right $\angle$s.

$\therefore\ AO$ passes through the centre of the ⊙ described about $BOCD$.

(*Solutions of Senate-House Problems for* 1878.)

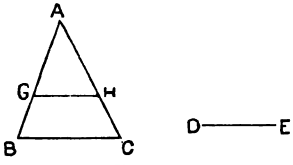


FIG. 492.

49. Let ABC be the △. Take DE a third proportional to AB, BC. Divide AB in G so that $AG : GB = AB : DE$, and draw $GH \parallel$ to BC.

Then $AG : GB =$ duplicate ratio of $AB : BC$;

$\therefore\ AG : GB =$ duplicate ratio of $AG : GH$;

$\therefore\ AG : GH = GH : GB$.

50. Let ABC be an equilateral △ inscribed in a ⊙.
Take D, E the middle pts. of arcs ADB, AEC.
Let DE cut AB, AC in the pts. P, Q.
Join DC, AE, DB.
Then ∵ $\angle CDE = \angle DCB$, subtended by an equal arc,

∴ DE is ∥ to BC.

Then ∵ $\angle DPB = \angle APE$,
and $\angle DBP = \angle AEP$,
and $DB = AE$, subtended by an equal arc;
∴ $DP = PA$, and similarly $EQ = QA$.
Hence, since APQ is an equilateral △,

$$PQ = DP = QE.$$

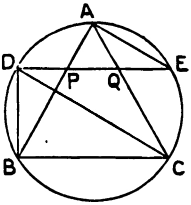


FIG. 493.

51. $\angle ACD = \frac{4}{3}$ of a right ∠;
∴ $\angle CAD = \angle CDA = \frac{1}{3}$ of a right ∠;
∴ $\angle BAD$ is a right ∠.
Also, $\angle ACE = \frac{1}{3}$ of a right ∠;
∴ $\angle AEC = \frac{1}{3}$ of a right ∠;
∴ $AE = AC$; and $\angle DAE$ is a right ∠;
∴ BAE is a straight line, and $BE = BD$;
∴ BED is an equilateral △, and ∴ $EC = DA$.
Then rect. DA, CE
= sq. on CE,
= sq. on BE − sq. on BC,
= 3 sq. on BC,
= 2 rect. BC, BC + sq. on BC,
= 2 rect. AC, AC + sq. on BC,
= rect. DE, AC + sq. on BC.

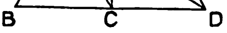


FIG. 494.

52. Since $CE : EA = BE : ED$,
∴ rect. CE, ED = rect. EA, BE;
∴ a ⊙ described round △ BCD will pass through A.
For, if not, let it cut AB in M.
Then rect. CE, ED = rect. BE, EM,
∴ $EM = EA$, which is absurd.

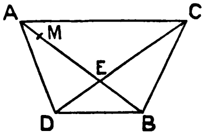


FIG. 495.

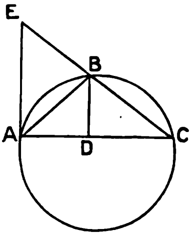


FIG. 496.

53. Let ABC be the $\triangle$, and AE the tangent at A.

Draw $BD \parallel$ to AE.

Now $\angle EAB = \angle BCD$,

and $\angle AEB = \angle DBC$;

$\therefore \triangle$s AEB, DCB are similar;

$\therefore AE : AB = CB : CD$.

Also $AC : AE = DC : DB$ (by similar $\triangle$s ACE, DCB),

$\therefore AC : AB = CB : DB$. (V. 21.)

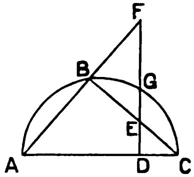


FIG. 497.

54. Let ABC be the $\triangle$, $DF \perp$ to AC.

Then $\angle BFE = \angle ECD$;

$\therefore \triangle$s ADF, EDC are similar;

$\therefore DF : DA = DC : DE$.

But $DA : DG = DG : DC$; (VI. 13.)

$\therefore DF : DG = DG : DE$. (V. 21.)

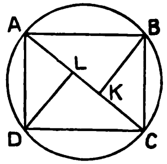


FIG. 498.

55. By VI. C,

rect. BK, diameter = rect. AB, BC;

rect. DL, diameter = rect. AD, DC;

$\therefore BK : DL =$ rect. AB, BC : rect. AD, DC.

56. Let ABC be the $\triangle$, and $DBFE$ the rectangle.

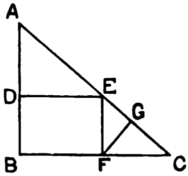


FIG. 499.

Draw $FG \perp$ to AC.

Then by similar $\triangle$s ADE, EGF,

$AD : AE = EG : EF$;

$\therefore$ rect. AD, $EF =$ rect. AE, EG, (1.)

and, by similar $\triangle$s ADE, FGC,

$DE : AE = GC : FC$;

$\therefore$ rect. DE, $FC =$ rect. AE, GC. (2.)

Hence from (1.) and (2.), observing that $BD = EF$, and $BF = DE$,

rect. AD, DB + rect. BF, $FC =$ rect. AE, EC.

57. Let OAB be an isosceles $\triangle$, and let the circle cut the base in C, D, and the sides in F, E.

Then $\because OE : OB = OF : OA$;

$\therefore FE$ is $\parallel$ to AB.

Hence if CE be joined, $\angle FEC = \angle ECD$;

and $\therefore$ arc $FC =$ arc ED;

$\therefore \angle FOC = \angle DOE$;

and $OA = OB$, $OC = OD$, and $\therefore AC = DB$. (I. 4.)

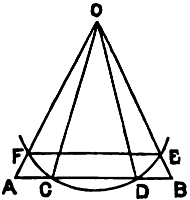


FIG. 500.

58. Let AC be the diameter through A, and let MP be drawn $\perp$ to AC produced.

Then the $\triangle$s ABC, APM are similar;

$\therefore AB : AC = AP : AM$;

$\therefore$ rect. $AB, AM =$ rect. AC, AP;

$\therefore$ rect. AC, AP is constant;

$\therefore P$ is a fixed point;

and the locus of M is a straight line through $P \perp$ to AP.

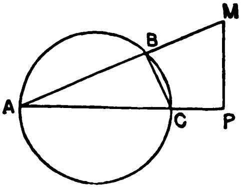


FIG. 501.

59. In the $\triangle ABC$ let AB be greater than AC.

Draw BE, $CF \perp$s on AC, AB.

Then $\because \triangle$s ABE, ACF are similar;

$\therefore AB : BE = AC : CF$;

$\therefore AB : AB - BE = AC : AC - CF$;

$\therefore AB : AC = AB - BE : AC - CF$;

$\therefore AB - BE$ is greater than $AC - CF$;

$\therefore AB + CF$ is greater than $AC + BE$.

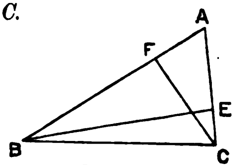


FIG. 502.

60. Let Ap, Aq, the $\perp$s from A on the external bisectors of the $\angle$s at B and C, meet BC produced in P and Q.

The $\triangle$s ABp, PBp are evidently equal,

$\therefore PB = AB$, and similarly, $CQ = AC$.

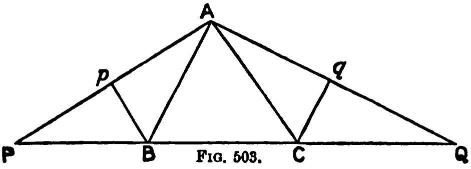


FIG. 503.

$\therefore PQ =$ the perimeter of $\triangle ABC$;

and as p and q are the middle points of AP, AQ;

$\therefore pq =$ half the perimeter of $\triangle ABC$.

61. Let A, B be fixed points on a $\odot$, and P a movable pt.; also let p be the centre of $\perp$s of $\triangle$ APB.

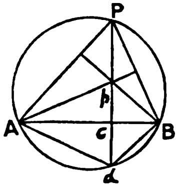


Fig. 504.

Let Pp produced meet AB in c, and the $\odot$ in a. Then $\angle ApB =$ supplement of $\angle APB$,

$$= \angle AaB.$$

$\therefore$ if $\triangle$ AaB be turned round AB, a will coincide with p;

$$\therefore ca = cp.$$

$\therefore$ $Pp = cP - ca =$ twice the distance of the centre of the $\odot$ from AB.

Hence Pp is constant for different positions of P, and being drawn in a constant direction, the $\triangle$s PQR, pqr must be equal, similar, and similarly placed, as one will coincide with the other when moved $\parallel$ to itself through a constant distance.

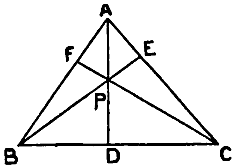


Fig. 505.

62. The $\triangle$s APF, ABD are similar;

$$\therefore AP : FP = AB : BD;$$

$\therefore$ rect. $AP, BD =$ rect. FP, AB.

Similarly, rect. $AP, CD =$ rect. EP, AC.

$\therefore$ rect. $AP, BC =$ rect. $FP, AB +$ rect. EP, AC.

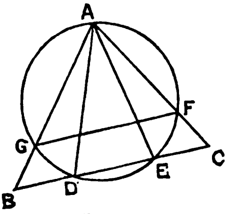


Fig. 506.

63. About $\triangle$ ADE describe a $\odot$, cutting AB, AC, produced if necessary, in G and F. Join FG.

Then $\because \angle GAD = \angle FAE$,

$\therefore$ arc $GD =$ arc EF;

$\therefore \angle GFD = \angle FDE$;

$\therefore$ FG is $\parallel$ to BC;

$\therefore AB : AC = BG : CF$;

$\therefore$ sq. on AB : sq. on $AC =$ rect. AB, BG : rect. AC, CF;

$=$ rect. BD, BE : rect. CD, CE.

(III. 36.)

64. Let A, B, C be the given points, and $l : m : n$ be the ratio of the perpendiculars.

Take a pt. D in AB so that $AD : BD = l : m$, and a pt. E in AC so that $AE : CE = l : n$.

Then DE is the line required, since $\perp$ s on it from A and B are as $AD : BD$, *i.e.* as $l : m$; and those from A and C are as $l : n$; and $\therefore$ those from B and C are as $m : n$.

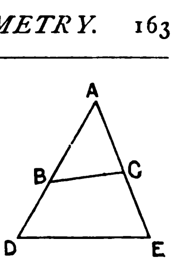


FIG. 507.

65. Let ABC, DEF, GHK be three similar $\triangle$ s.

Take MN a fourth proportional to BC, EF, HK; and on MN describe a $\triangle\ LMN$ similar to $\triangle\ ABC$.

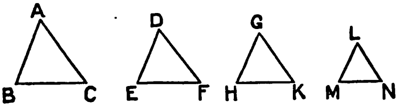


FIG. 508.

Then, by VI. 24,

$$\triangle ABC : \triangle DEF = \triangle GHK : \triangle LMN.$$

66. Let ABC be a $\triangle$, DEF the middle pts. of its sides.

Let the bisectors of the sides meet in O.

Join FE.

Then AOB, EOF are similar $\triangle$ s;

$\therefore AO : OE = AB : EF,$

$= 2 : 1$

Similarly, $BO : OF = 2 : 1$, and $CO : OD = 2 : 1$.

Draw $AN \perp$ to BC, and $OP \perp$ to AN.

Then $\triangle ABC : \triangle OBC = AN : PN,$

$= AE : OE,$

$= 3 : 1.$

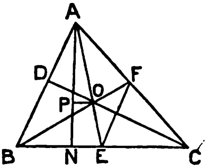


FIG. 509.

67. Let OP, OQ be the given lines, C a pt. such that if CP, CQ be drawn $\perp$ to OP, OQ, the line PQ is constant. Then since PQ is constant, and $\angle\ POQ$ is fixed, the circle circumscribing OPQ is of constant diameter, and OC is the diameter of this $\odot$. Hence the locus of C is a $\odot$ with centre O.

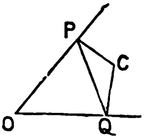


FIG. 510.

68. Let $EFGH$ be one of the ▱s, and let AD produced meet CH produced in M. Let DB, EG intersect in O.

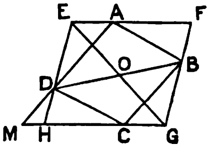


FIG. 511.

Then $\angle BCG = \angle DMH$,

$= \angle EAD$;

and $\angle AED = \angle BGC$, and $AD = BC$;

$\therefore ED = BG$, and, since △s EOD, BOG are equiangular,

$\therefore OB = OD$ and $OE = OG$;

$\therefore O$ is the point of bisection of diagonals of $ABCD$, $EFGH$.

Hence the diagonals of all the ▱s pass through O.

69. Join RT.

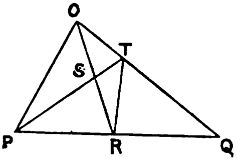


FIG. 512.

$$\text{Then } \triangle ORQ = \triangle OPR;$$
$$\text{and } \triangle QRT = \triangle PTR;$$
$$\therefore \triangle ORQ - \triangle QRT = \triangle OPR - \triangle PTR;$$
$$\therefore \triangle ORT = \triangle OSP - \triangle STR;$$
$$\therefore 2\triangle STR = \triangle OSP - \triangle STR;$$
$$\therefore 3\triangle STR = \triangle OSP;$$
$$\therefore 6\triangle STR = 2\triangle OSP,$$
$$= \triangle OPR,$$
$$= \triangle ORQ;$$
$$\therefore 3\triangle ORT = \triangle ORQ;$$
$$\therefore OQ = 3OT.$$

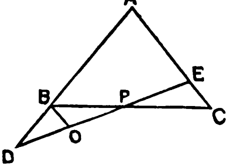


FIG. 513.

70. Let P be the middle pt. of BC.

Draw $BO \parallel$ to AC, and meeting EPD in O.

Then $\because BP = PC$, $\therefore OB = EC$;

and, since BO is $\parallel$ to AC,

$$AD : BD = AE : BO,$$
$$= AE : EC.$$

71. Area of △ ABC

$=\frac{1}{2}$ rect. CP, AB,

$=\frac{1}{2}$ rect. $CP, 2AQ$,

$=$ rect. CP, CQ,

$=$ a constant.

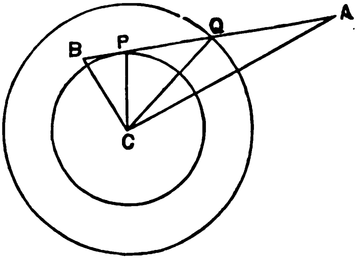


FIG. 514.

72. Let PD meet the concave ○ce in R.

Then since CAP, CBP are rt. ∠s, a ⊙ described on PC as diameter passes through A and B. And since PFC is a rt. ∠, F is a pt. in the ○ce of this ⊙.

Then rect. $FP, FE=$ rect. $FE, EP+$ sq. on EF, (II. 3.)

$=$ rect. $AE, EB+$ sq. on EF, (III. 35.)

$=$ rect. $RE, ED+$ sq. on EF, (III. 35.)

$=$ sq. on FD. (II. 5.)

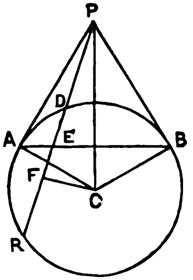


FIG. 515.

73. Let α, β, γ be the points of contact of the inscribed ⊙, and let the three ⊙s be drawn as described.

Then $A\gamma$ and $A\beta$ are equal, being tangents to the inscribed ⊙. Join AG; it must be a tangent to the two ⊙s $G\gamma$, $G\beta$, since the rectangle under the segments of any line drawn to these ⊙s from A must be equal to the squares on $A\gamma$, $A\beta$, which are equal, and therefore AG cannot meet these ⊙s in any other point than G.

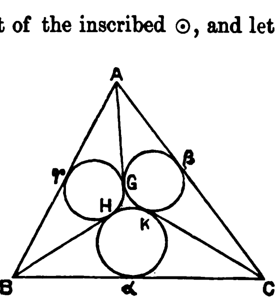


FIG. 516.

Similarly with regard to BH, CK.

But tangents to the three ⊙s at G, H, K meet in a point;

$\therefore$ AG, BH, CK meet in a point.

74. $BH : BK = EH : AK,$

$= FL : AK,$

$= CL : CK;$

$\therefore$ rect. BH, CK = rect. CL, BK.

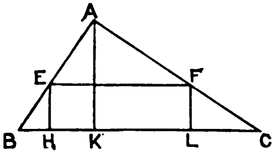


FIG. 517.

Now sq. on AC = rect. BC, CK = rect. BH, CK + rect. CH, CK;
and sq. on AB = rect. BC, BK = rect. BL, BK + rect. CL, BK;
$\therefore$ sq. on AC − sq. on AB = rect. CH, CK − rect. BL, BK.

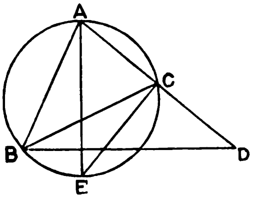


FIG. 518.

75. Let AE be the diameter; join BC, CE.

Then $\angle ABD + \angle BAE$ = a rt. $\angle$;
and $\angle ACB + \angle BCE$ = a rt. $\angle$;
also $\angle BAE = \angle BCE$ in the same segment;

$\therefore \angle ABD = \angle ACB.$

Hence $\triangle$s ABD, ACB are similar,

$\therefore AD : AB = AB : AC.$

76. Let ABC be the fixed chord, and APQ the movable chord, and let PB, QC intersect in R.

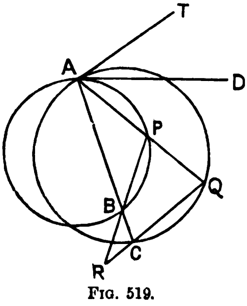


FIG. 519.

Draw TA, DA tangents to the ⊙s at A.

Then $\angle TAQ = \angle ACQ$ in alternate segment;

and $\angle DAP = \angle ABP$ in alternate segment;

$\therefore \angle TAD = \angle ACQ - \angle ABP,$

$= \angle ACQ - \angle RBC = \angle BRC.$

Now $\angle TAD$ is constant, and $\therefore \angle BRC$ is constant.

Hence the locus of R is a circle passing through BC.

77. $AE:ED=BC:CD$. (1.

Also, $AE:DA=BC:BD$,

$=AB:BD$,

$=EC:CD$; (2.)

∴, compounding the ratios in (1.) and (2.),

sq. on AE : rect. ED, DA = rect. BC, CE : sq. on CD.

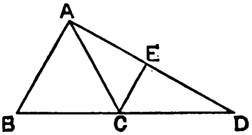


FIG. 520.

78. Bisect BC in E; then E is the centre of the $\odot$ described about $\triangle ABC$.

Draw $ED \parallel$ to BA, then ED bisects AC.

Then $\angle AEB = 2\angle ACE$. (III. 20.)

Now AE is greater than AD;

∴ AE is greater than AB;

∴ $\angle ABE$ is greater than $\angle AEB$;

∴ $\angle ABE$ is greater than $2\angle ACE$.

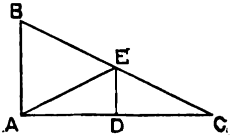


FIG. 521.

79. Describe a $\odot$ about the $\triangle ABC$, and produce AB to D, making $BD=AB$.

From D draw $DE \parallel$ to BC, meeting the ○ce in E.

Join AE, cutting BC in F.

Then ∵ $AB=BD$, ∴ $AF=FE$.

Now rect. BF, FC = rect. AF, FE = sq. on AF;

∴ $BF:AF=AF:FC$.

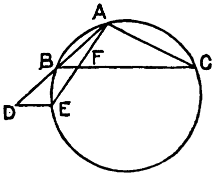


FIG. 522.

80. Since $\triangle$s FAG, DAE are similar,

∴ $AG:AE=FA:AD$;

and since $\triangle$s FAC, DAB are similar,

∴ $FA:AD=CA:AB$.

Hence $AG:AE=CA:AB$;

∴ EG is $\parallel$ to BC.

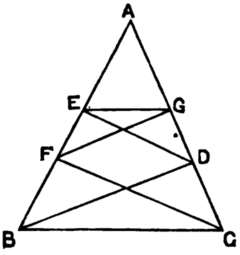


FIG. 523.

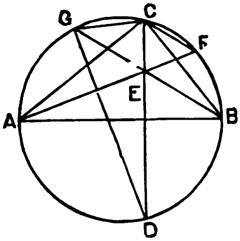


FIG. 524.

81. $\angle AFC = \angle ABC$, in same segment;
$\angle AFD = \angle ACD$, in same segment,
$= \angle ABC$, because CD cuts AB at rt. $\angle$s;
$\therefore \angle AFC = \angle AFD$.
Again, $\angle DGB = \angle DCB = \angle CAB = \angle CGB$;
$\therefore \angle$s at F and G are bisected by AF, BG;
$\therefore CF : FD = CE : ED$, (VI. 3.)
$= CG : GD$.

82. Let p, f, g be the pts. of tangency.

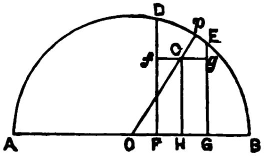


FIG. 525.

Then sq. on CH
$=$ sq. on $OC -$ sq. on OH,
$= (OC + OH).(OC - OH)$, (II. B. p. 84.)
$= (OC + OF + FH).(OC - OG + HG)$.
Now $FH = HG = fC = gC = Cp$;
and $OC + Cp = Op = OA = OB$;
$\therefore$ sq. on $CH = (OA + OF).(OB - OG)$,
$=$ rect. AF, BG.

83. Let BC be the given st. line. Divide BC in D, so that $BD : DC$ is in the given ratio. On BC describe the $\triangle$ ABC, such that $BA : AC$ is in the given ratio. Produce BC to E, so that $BE : EC = BD : DC$.

Then $\angle DAE$ is a rt. $\angle$; (p. 251, Ex. 3.)
and $\therefore$ a $\odot$ described on DE as diameter passes through A.
Let F be the centre of this circle, and G any pt. in the ○ce.

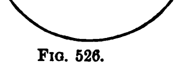


FIG. 526.

Then $\angle FAD = \angle FDA$;
$\therefore \angle FAC + \angle CAD = \angle ABD + \angle BAD$.
Now $\angle CAD = \angle BAD$; (VI. 3.)
$\therefore \angle FAC = \angle ABD$;
$\therefore \triangle$s ABF, ACF are similar;
$\therefore BF : FA = FA : FC$;
or, $BF : FG = FG : FC$;
$\therefore \angle CGF = \angle CBG$. (VI. 6.)
But $\angle FGD = \angle FDG$;
$= \angle CBG + \angle BGD$;
$\therefore \angle FGD - \angle CBG = \angle BGD$;
$\therefore \angle FGD - \angle CGF = \angle BGD$;
$\therefore \angle DGC = \angle BGD$;
$\therefore BG : GC = BD : DC$.

84. Let OA, OB be the tangents, and OCD the secant.

Since $\angle\ OAC = \angle\ ADC$, in alternate segment,

$\therefore\ \triangle$s ADO, OAC are equiangular;

$\therefore\ AD : AC = DO : AO.$

Similarly $\triangle$s ODB, BCO are equiangular, and

$\therefore\ DB : BC = DO : OB,$

$= DO : AO;$

$\therefore\ AD : AC = DB : BC;$

$\therefore$ rect. AD, BC = rect. AC, DB.

(M'DOWELL'S *Exercises.*)

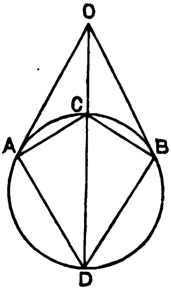


FIG. 527.

85. Let ABC, BDC be triangles on the same base, and let $\angle\ BAC = \angle\ BDC$.

Then a circle described about $\triangle\ ABC$ will pass through D.

Let d = diameter of this ⊙.

Draw AM, DM $\perp$s to BC.

Then area of $\triangle\ ABC = \frac{1}{2}\ AM \cdot BC$;

and area of $\triangle\ DBC = \frac{1}{2}\ DN \cdot BC$;

$\therefore$ area of $\triangle\ ABC$: area of $\triangle\ DBC = AM : DN$,

= rect. AM, d : rect. DN, d,

= rect. BA, AC : rect. BD, DC.

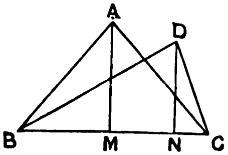


FIG. 528.

86. Let P be a pt. such that

$AP : PB = AC : CB.$

Then CP bisects $\angle\ APB$; (VI. 3.)

and if AP be produced to Q,

DP bisects $\angle\ BPQ$; (VI. A.)

$\therefore\ CP$ is $\perp$ to DP;

$\therefore$ a semicircle described in CD passes through P.

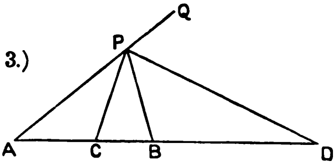


FIG. 529.

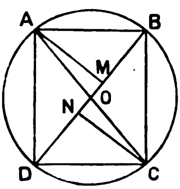


FIG. 530.

87. Let $ABCD$ be the quadrilateral, O the pt. of intersection of the diagonals.

Then if the diagonals are at rt. ∠s,

$$\text{area of quadrilateral} = \tfrac{1}{2}\, AC \times BD.$$

If they are not at rt. ∠s, draw AM, CN ⊥ s to BD: then area of quadrilateral $= \frac{1}{2}(AM + CN) \times BD$.

Now AM is less than AO, and CN is less than CO;

$\therefore$ $AM + CN$ is less than AC.

88. It is clear that if we prove the proposition for any one triangle, it will be true for all triangles having their sides parallel to the sides of the particular triangle.

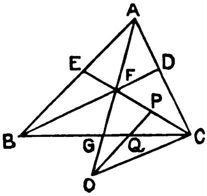


FIG. 531.

Now AF produced bisects BC in G, and $AF = 2 . FG$.

Draw $CO \parallel$ to BF, then FCO is the particular triangle to which we have referred.

Since $AD = DC$, $\therefore AF = FO$, and $\therefore GO = GF$, and the line CG bisecting FO coincides with BC, and $\therefore$ is $\parallel$ to it.

Again, if OQP bisect FC,

$CP = \frac{1}{3}$ of CE, and $CQ = \frac{2}{3}$ of $CG = \frac{1}{3}$ of CB;

$\therefore$ OP is $\parallel$ to AB.

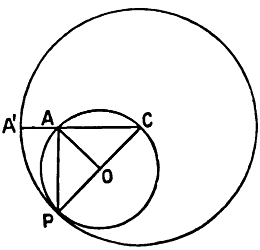


FIG. 532.

89. Let C be the centre of the larger circle, and O the centre of the smaller ⊙.

Draw CAA' to meet the larger ⊙ in A'.

Then $\angle AOP = 2 \angle A'CP$;

$\therefore$ arc AP = arc $A'P$;

$\therefore$ A' is the original position of A, and PA being ⊥ to CA' is constant in direction.

90. Join FH and EK. Then since $\angle EAK = \frac{2}{3}$ of a rt. $\angle$,

$\therefore \triangle EAK$ is equilateral, and $\therefore EK$ is $\parallel$ to CB;
and similarly FH is $\parallel$ to AC.

Then in $\triangle$s GEA, FEC, $\because \angle GEA = \angle FEC$,
and $\angle EAG =$ supplement of $\angle ECF$,

$$\therefore GE : EF = GA : CF. \qquad (1.)$$

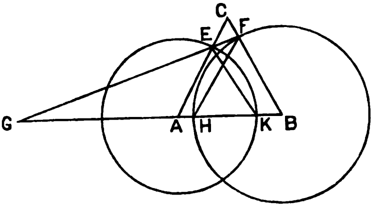


FIG. 533.

And in $\triangle$s GEK, EFC, $\because \angle GKE = \frac{2}{3}$ of a rt. $\angle = \angle ECF$,
and $\angle GEK = \angle GFB =$ supplement of $\angle EFC$,

$$\therefore GE : EF = GK : CE. \qquad (2.)$$

From (1.) and (2.) $GA : GK = CF : CE$.
The other result follows similarly.

91 Let OD, OE, OF be the perpendiculars on BC, AC, AB.

Join OB, OA, DF, FE.

Circles can be described about $DOEC$, $OEAF$, and $OFDB$;

$\therefore \angle BOD = \angle BFD$, and $\angle AOE = \angle AFE$.

Now $\angle DOE + \angle DCE = 2$ rt. $\angle$s,
$= \angle BOA + \angle DCE$;
$\therefore \angle DOE = \angle BOA$;
$\therefore \angle AOE = \angle BOD$;

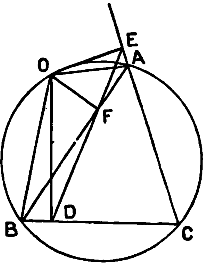


FIG. 534.

Hence $\angle BFD = \angle AFE$; and $\therefore$, since BFA is a straight line, DF and FE must be in the same straight line.

(M'DOWELL'S *Exercises.*)

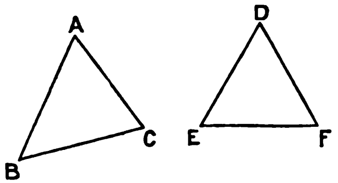


FIG. 535.

92. Let ABC, DEF be the two triangles, such that $\angle BAC = \angle EDF$, and $ED = FD$; and suppose that

$$AB : DE = DE : AC.$$

Then $AB : DE = DF : AC$; and $\therefore$ by VI. 15,

$$\triangle ABC = \triangle DEF.$$

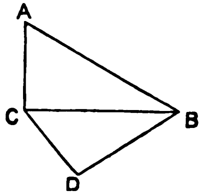


FIG. 536.

93. Since the $\triangle$s are similar,

$$AB : BC = BC : BD;$$

$\therefore AB : BD =$ duplicate ratio of $AB : BC$,

$$= \triangle ABC : \triangle BCD.$$

94. Describe any equilateral $\triangle ABC$.

Take D a point of trisection of BC, and in AD take $AO =$ the given straight line.

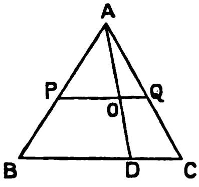


FIG. 537.

Draw $POQ \parallel$ to BC, then $\triangle APQ$ is the triangle required.

For it is equilateral, since its $\angle$s are equal to those of $\triangle ABC$, each to each, and O is a point of trisection of PQ, for

$$AD : DO = BD : PO,$$

$$\text{and } AD : DO = DC : OQ,$$

$$\therefore BD : DC = PO : OQ; \therefore PO = 2OQ.$$

95. Let d be the point through which the lines pass.

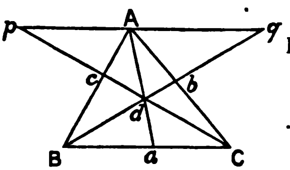


FIG. 538.

Draw $pAq \parallel$ to BC, meeting Bb, Cc produced in q and p.

Then $Aq : BC = Ab : Cb$,

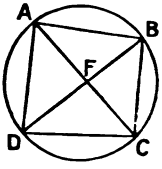
and $BC : Ap = Bc : Ac$;

$\therefore Aq : Ap =$ rect. Ab, Bc : rect. Cb, Ac.

Now $Aq : Ap = Ba : Ca$;

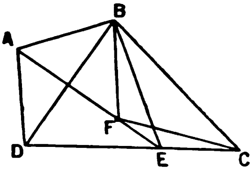
$\therefore Ba : Ca =$ rect. Ab, Bc : rect. Cb, Ac;

$\therefore Cb : Ca =$ rect. Ab, Bc : rect. Ba, Ac.



96. Since $\triangle AFD : \triangle AFB = FD : BF$,
$= \text{rect. } AF, FD : \text{rect. } AF, BF$;
and $\triangle AFB : \triangle BFC = AF : FC$,
$= \text{rect. } AF, BF : \text{rect. } BF, FC$;
$\therefore \triangle AFD : \triangle BFC = \text{rect. } AF, FD : \text{rect. } BF, FC$.
But $\triangle AFD : \triangle BFC = \text{sq. on } AD : \text{sq. on } BC$;
$\therefore \text{rect. } AF, FD : \text{rect. } BF, FC = \text{sq. on } AD : \text{sq. on } BC$.

FIG. 539.



97. Since $\angle BAF = \angle BDC$, in same segment of $\odot ABED$,
and $\angle AFB =$ supplement of $\angle BFE$,
$= \angle DCB$, by $\odot BFEC$;
$\therefore \triangle$s ABF, DBC are similar.
Again, $\angle BFC = \angle BEC =$ supplement of $\angle BED = \angle BAD$,
and $\angle BCF = \angle BEF = \angle BDA$;
$\therefore \triangle$s BCF, ADB are similar.

FIG. 540.

98. Draw $EF \parallel$ to AB, meeting AC in E, and BC in F.
Draw $CO, EQ, FP \parallel$ to BD.
Then O is the middle pt. of AB; (VI. 2.)
and $OP : PB = CF : FB$,
$= CE : EA$,
$= OQ : QA$;
$\therefore OP + PB : PB = OQ + QA : QA$;
$\therefore OB : PB = OA : QA$;
$\therefore PB = QA$.

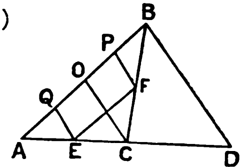


FIG. 541.

99. Since $CF : CD = EF : BD$,
and $CD : DF = AC : EF$,
$\therefore CF : DF = AC : BD$;
$\therefore CF : AC = DF : BD$;
and $\angle ACF = \angle BDF$;
$\therefore \angle AFC = \angle BFD$; (VI. 6.)
$\therefore \angle AFE = \angle BFE$.

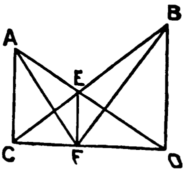


FIG. 542.

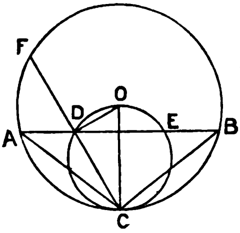


FIG. 543.

100. Let CD produced meet the larger $\odot$ in F.

Then $\angle ODC$ is a rt. $\angle$, and
$\therefore DF=DC$.
Now rect. AD, DB = rect. FD, DC,
= sq. on DC.
Similarly, rect. AE, EB = sq. on CE.

101. Describe a $\odot$ about the $\triangle ABC$.

Produce EB to meet the ○ce in F, and join AF.

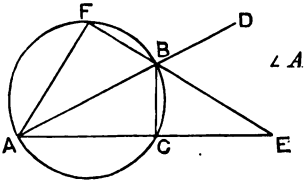


FIG. 544.

Then $\because \angle EBC = \angle EBD$,
$= \angle FBA$,
and
$\angle AFB$ = supplement of $\angle BCA = \angle BCE$,
$\therefore$ the $\triangle$s EBC, FBA are equiangular;
$\therefore AB : BF = EB : BC$;
$\therefore$ rect. AB, BC = rect. BF, EB;
$\therefore$ rect. AB, BC + sq. on BE
= rect. BF, EB + sq. on BE,
= rect. FE, EB,
= rect. EA, EC.

102. In the $\triangle$s QAB, PCB, since $\angle QBA = \angle PBC$, and $\angle QAB$ is the supplement of $\angle PCB$,

$\therefore QA : QB = PC : PB$.

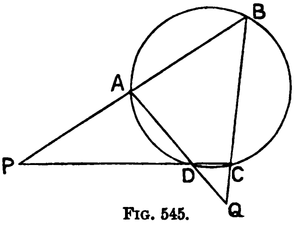


FIG. 545.

Also $\angle P + \angle Q = 4$ rt. $\angle$s $- (\angle DAB + \angle BCD + 2\angle ABC)$,
$= 2$ rt. $\angle$s $- 2\angle ABC$;
$\therefore \frac{1}{2}(\angle P + \angle Q)$ = complement of $\angle ABC$.

103. On the given diameter describe a ⊙, and from it cut off a segment *ACB* containing an ∠ equal to the given vertical ∠.

Divide *AB* in *D*, so that *AD* : *DB* in the given ratio of the sides. Draw the diameter *EF* ⊥ to *AB*.

Join *ED* and produce it to meet the ○ce in *C*.

Then since arc *AE*=arc *EB*, the ∠ *ACB* is bisected by *CE*;

$\therefore AC : CB = AD : DB,$
$\quad = \text{the given ratio.}$

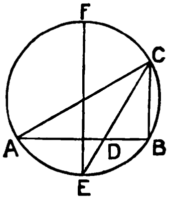


FIG. 546.

104. Since the distance *OP* is known, *OC* is known, and we can find a square=difference of squares on *OC*, *CT*, the side of this square gives *OT*.

Join *OC*, and what is required is done.

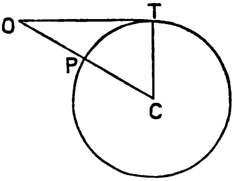


FIG. 547.

105. Let *A*, *B*, and *C* be the three points.

Divide *AB* in *D* so that *AD* : *DB*=*p* : *q*, and in *AB* produced take a point *E*, such that *AE* : *BE*=*p* : *q*, and upon *DE*, as a diameter, describe a circle. Every point on this circle has its distances from *A* and *B* proportional to *p* and *q* respectively. (See Ex. 83.)

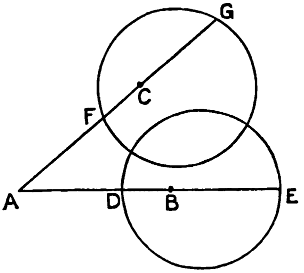


FIG. 548.

Describe a similar circle relative to *A* and *C*.

The points of intersection of these circles, when intersection is possible, satisfy the required condition.

106. Since AD is bisected at C, $\angle ABC = \angle CBD$;

$\therefore \angle ABC = \angle CAD$, and $\therefore$ a $\odot$ passing through AEB touches AC.

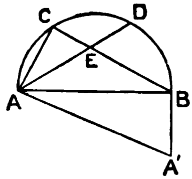


FIG. 549.

Draw $AA' \perp$ to CA, and $BA' \perp$ to AB.

Then AA' must be a diameter of the $\odot$ passing through AEB,

and $\angle A'AB =$ complement of $\angle CAB$,

$= \angle CBA$;

$\therefore \triangle A'AB$ is similar to $\triangle ABC$;

$\therefore BC : AB = AB : AA'$.

107. Draw $AT \perp$ to the given line TP, and take B in AT, such that rect. $AB, AT =$ rect. AQ, AP.

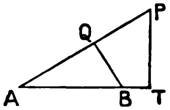


FIG. 550.

Then $AB : AQ = AP : AT$.

Hence $\triangle$s BAQ and PAT are similar;

and $\therefore \angle AQB = \angle ATP =$ a rt. $\angle$;

$\therefore$ the locus of Q is a $\odot$ having AB for its diameter.

108. We can describe a rectangle having for one side the sum of the given lines, and for the other the difference. The area of this rectangle will be the difference of the squares on the lines. We can then describe on the given line a rectangle equal to this rectangle, and what was required is done.

109. Let $C'B$ and $B'C$, the external bisectors of the angles at B and C, meet in A'.

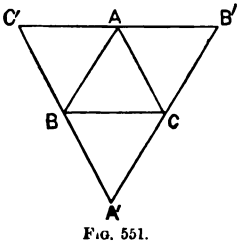


FIG. 551.

First, since $\angle A' =$ supplement of $2 \angle A$, $\angle A'$ cannot $= \angle A$ unless $\angle A = \frac{1}{3}$ of 2 rt. $\angle$s, in which case the $\triangle$s are equilateral. Hence $A'B'C'$ is not similar to ABC.

Next, to show that $A'B'C'$ cannot be similar to BCA.

If so, 2 rt. $\angle$s $- 2 \angle A = \angle B$,

and 2 rt. $\angle$s $- 2 \angle B = \angle C$,

and 2 rt. $\angle$s $- 2 \angle C = \angle A$.

Hence we get $\angle A - \angle C = O$,

and $\angle A - \angle B = O$.

$\therefore$ the $\triangle$s are equilateral.

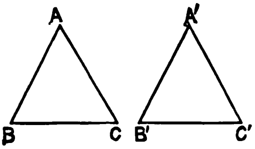


FIG. 552.

110. Area of ABC : area of $A'B'C'$
$=$ rect. AB, AC : rect. $A'B', A'C'$,
$=$ rect. $A'C', AC$: rect. $A'B', A'C'$,
$= AC : A'B'$.

111. Let A, B, C, D, E, F be the angular pts. of a regular hexagon. Describe a ⊙ about it.

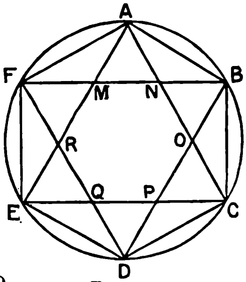


FIG. 553.

Join AE, EC, CA ; BF, FD, DB.

Then △s AEC, BDF are equilateral.

And since FB, EC cut off equal arcs, they are ||.

∴ $AM = AN$, and △ AMN is equilateral.

Now in △s ABM, FEM,

∵ $AB = FE$, and ∠ AMB = ∠ FME,
and ∠ ABM = ∠ FEM, in the same segment,
∴ $AM = FM$; and similarly $AN = BN$;
∴ $FM = MN = NB$;
∴ △ $FRM = \frac{1}{9}$ of △ BDF = △ BON = △ QPD ;
∴ hexagon $MNOPQR = \frac{2}{3}$ of △ BDF,
$= \frac{2}{3}$ of $\frac{1}{2}$ of hexagon $ABCDEF$, (p. 202, IV. 15.)
$= \frac{1}{3}$ of hexagon $ABCDEF$.

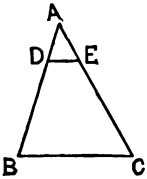


FIG. 554.

112. Let ABC be the △, and DE the line || to BC.

Then △ $ABC = 9$ times △ ADE ;
∴ AC = three times AE ;
∴ D, E are points of trisection of AB, AC.

113. Let APQ be the chord to the circles from A.
Join CP, and draw DR to R the middle pt. of PQ.
Then CP, DR being both ⊥ to AQ are ||.

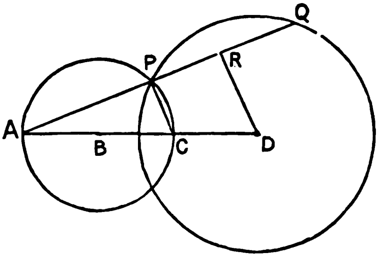


FIG. 555.

$$\therefore AP : PR = AC : CD;$$
$$\therefore AP = 2\,PR;$$
$$\therefore AP = PQ.$$

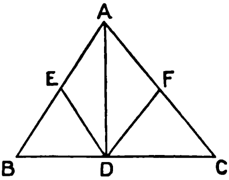


FIG. 556.

114. Since $EA = EB = ED$, a ⊙ described with centre E and radius EA passes through D and B;

$\therefore \angle ADB$ is a rt. ∠;

$\therefore ADC$ is a rt. ∠;

∴ a ⊙ described with centre F and radius FA passes through D.

Next, let AEF be an acute-angled △. Produce AE, AF to B and C, so that $EB = AE$, and $FC = AF$.

Then BC is || to EF.

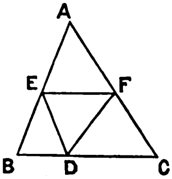


FIG. 557.

Turn △ AEF over round EF, then A will fall on a pt. D in BC such that $ED = EA$, and $FD = FA$; and ∴ △ EAF will coincide with △ EDF.

$$\text{Then } \angle EAF + \angle AEF + \angle AFE$$
$$= \angle EDF + \angle FED + \angle EFD,$$
$$= \angle EDF + \angle EDB + \angle FDC,$$
$$= 2 \text{ rt. } \angle\text{s}.$$

115. Since $\triangle BAC = \triangle DAE$,
$\therefore$ $\triangle BCE = \triangle DCE$;
and $\therefore$ CE is $\parallel$ to BD;
$\therefore$ $BC : BF = DE : DF$;
and $\therefore$ $\triangle BCA : \triangle BFA = \triangle DEA : \triangle DFA$.
But $\triangle BCA = \triangle DEA$;
$\therefore$ $\triangle BFA = \triangle DFA$;
and $\therefore$ perpendiculars from B and D on FA are equal;
and $\therefore$ FA bisects BD and its parallel CE.

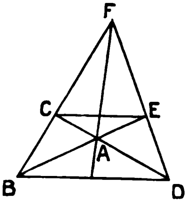


FIG. 558.

116. Sq. on PN = rect. AN, NB,
$= \frac{1}{6} \cdot \frac{5}{6}$ of sq. on AB;
sq. on QN = rect. AN, NO,
$= \frac{1}{6} \cdot \frac{1}{3}$ of sq. on AB,
$\therefore$ sq. on PN : sq. on $QN = 5 : 2$.

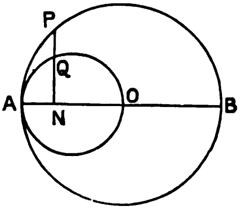


FIG. 559.

117. First describe a rectangle equal to the sum of the given square and the given rectangle. Then describe a square equal to this rectangle.

Miscellaneous Exercises on Book XI.

Page 334.

1. Let OA, OB, straight lines in the plane AOB, be equally inclined to the plane ACB.

Let AB be the common section of the planes. From O draw $OC \perp$ to the plane ACB, and join AC, BC.

Then $\because$ $\angle OAC = \angle OBC$, by hypothesis, and $\angle OCB = \angle OCA$, each being a rt. $\angle$, and OC is common to the $\triangle$s AOC, BOC,
$\therefore$ $OA = OB$,
and $\therefore$ $\angle OAB = \angle OBA$.

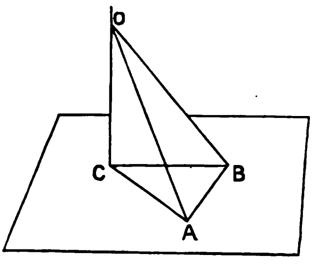


FIG. 560.

2. The two planes $ACBX$, $ACBY$ are at rt. ∠s.

In plane $ACBX$ from the pt. C in the intersection AB of the planes, CE, CF are drawn making ∠ ACE = ∠ BCF. DCD' is any straight line drawn through C in the plane $ACBY$. Take $CE = CF$.

Draw EA, FB ⊥ to AB.

Then ∵ $CE = CF$, and ∠ CAE = ∠ CBF, and ∠ ECA = ∠ BCF,

∴ $AC = BC$, and $AE = BF$.

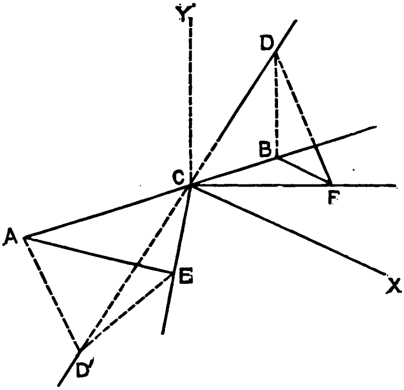


FIG. 561.

Again, draw in plane $ACBY$ the lines AD', BD ⊥ to AB, and meeting DCD' in D' and D.

Then ∵ $AC = BC$, and ∠ ACD' = ∠ BCD, and ∠ CAD' = ∠ CBD,

∴ $CD' = CD$, and $AD' = BD$.

Join $D'E$, DF, and then

∵ $AD' = BD$, and $AE = BF$, and ∠ $D'AE$ = ∠ DBF,

∴ $D'E = DF$; and ∴ in the △s CED', CFD,

∵ $CD' = CD$, and $CE = CF$, and $D'E = DF$,

∴ ∠ $D'CE$ = ∠ DCF.

3. Draw AF ⊥ to BC, and BG ⊥ to AC, and let these ⊥s intersect in D. Join EF.

Then ∵ DE is ⊥ to plane ABC, every plane through DE is ⊥ to plane ABC, and $\therefore$ the plane through DE and AF is $\perp$ to plane ABC, and AF is the intersection of the planes. Now FB is drawn in plane ABC, and is $\perp$ to the section AF, and $\therefore$ is $\perp$ to every line drawn in the plane EAF, and EA is in this plane, and $\therefore BC$ is $\perp$ to EA. Similarly, BE is $\perp$ to AC, and EC is $\perp$ to AB.

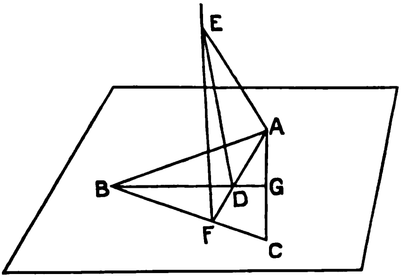


FIG. 562.

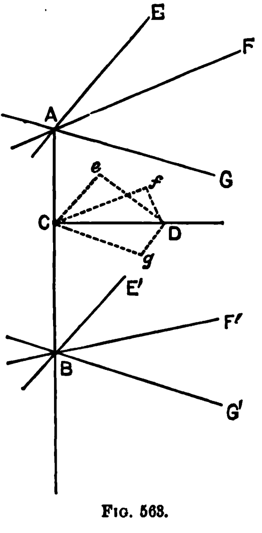


FIG. 563.

4. Let AB be the common intersection of a number of planes $EABE'$, $FABF'$, $GABG'$ . . . and D the given point. Through D draw a plane $DCefg \perp$ to the section AB, and cutting it in C; then since ACB is $\perp$ to the plane $DCefg$, every plane through ACB is $\perp$ to $DCefg$, and $\therefore DCefg$ is $\perp$ to all the given planes having the common section AB, and $\therefore$, if De, Df, Dg . . . be $\perp$s from D on these planes, e, f, g . . . all lie in the same plane through $DC \perp$ to AB, and $\therefore$ since CD is fixed, and the $\angle$s made by joining C to e, f, g . . . are all rt. $\angle$s, the locus of feet of $\perp$s is a circle on DC as diameter.

5. Let $EABE'$, $FABF'$ (as in diagram to Ex. 4) be the given planes. Let D be the pt. from which the $\perp$ s on these planes are drawn. Through D draw a plane $DCef \perp$ to AB, the section of the given planes, and draw De, Df $\perp$ s from D on these planes.

Then $\angle\, eCf =$ angle of inclination of planes $EABE'$, $FABF'$.

And since De is $\perp$ to Ce, and Df is $\perp$ to Cf,

$$\therefore \angle\, eDf = \angle\, eCf.$$

6. Let $AH'GH$ be the plane through $A \perp$ to plane $AFF'A'$, having the line HGH' its intersection with the plane $HFH'F'$.

Now since AF, $A'F'$ are two $\perp$ s to the plane $HFH'F'$, this plane is $\perp$ to the plane drawn through AF, $A'F'$, and also plane $AH'GH$ is $\perp$ to the plane through AF, $A'F'$.

$\therefore$ the two planes $FHH'F'$, $AH'GH$, intersecting in the line HGH', are each of them $\perp$ to the plane $AFF'A'$, and $\therefore$ the line HGH' is $\perp$ to the plane $AFF'A'$, and $\therefore$ $\perp$ to every straight line drawn through G in that plane, and $\therefore$ $\perp$ to FGF'.

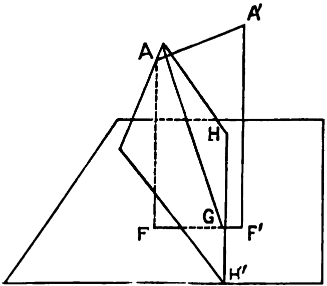


Fig. 564.

7. Let FB, FC be equal straight lines drawn from the pt. F to the plane ECB. Draw $FE \perp$ to the plane ECB, and join EB, EC.

Then FE is $\perp$ to EB and also to EC.

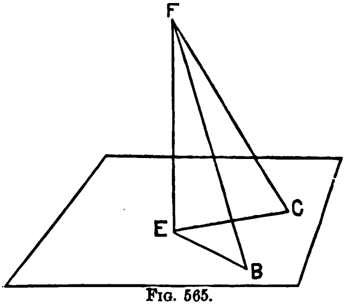


Fig. 565.

$\therefore$ in the right-angled Δ s FEB, FEC, $FB = FC$, and FE is common, and $\therefore \angle\, FBE = \angle\, FCE$, that is, FB and FC are equally inclined to *the plane.*

8. This exercise is the same as Ex. 5.

9. From A draw $AB \perp$ to the plane BDE, and $AD \perp$ to the line DE in that plane.

Make $DE = AB$, and join AE, BE, BD.

Then in $\triangle$s ABE, ADE,

$\because$ $AB = DE$, and AE is common, and rt. $\angle ABE =$ rt. $\angle ADE$;

$\therefore$ $AD = BE$.

Then in $\triangle$s ABD, BDE,

$\because$ $AD = BE$, and BD is common, and $AB = ED$;

$\therefore$ $\angle ABD = \angle BDE$;

$\therefore$ $\angle BDE$ is a right angle.

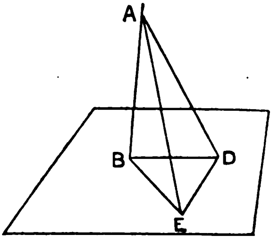


FIG. 566.

10. The angles containing a solid angle are together less than four rt. $\angle$ s.

$\therefore$ if equilateral $\triangle$ s alone be used, a solid angle can be made by 5, 4, or 3 equilateral triangles. If 6 were employed, the solid $\angle$ would become equal to four rt. $\angle$s, and $\therefore$ the bounding lines would all be in one plane. If only squares be used, there is only one way of forming a solid $\angle$, that is by using 3, for if 4 were used, the solid $\angle$ would become equal to four rt. $\angle$s. Taking *one* square, we can form a solid $\angle$ either with 4, 3, or 2 equilateral $\triangle$ s.

Taking *two* squares, we can form a solid $\angle$ with 2 equilateral $\triangle$ s only, for if 3 were used, the solid $\angle$ would become equal to four rt. $\angle$ s.

$\therefore$ the total number of ways is $3 + 1 + 3 + 1$, or, 8.

11. Let AB be the intersection of the two planes, inclined at a given $\angle$.

Let BCD be a plane drawn $\perp$ to AB, and having BC, BD for its intersections with the given planes. Join any two pts. C and D, one in each of these intersections. With CD as diameter describe a sphere cutting AB in E, and join CE, DE. Then the plane through C, D, E intersects the two given planes in CE, DE, and these lines are at right angles, since CD is the diameter of a sphere, and E is a point in the circumference of the sphere.

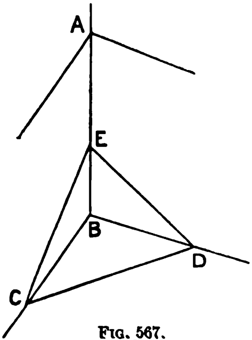


FIG. 567.

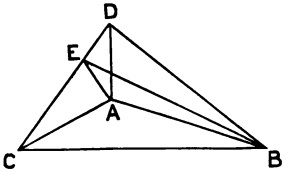


FIG. 568.

12. Since AB is $\perp$ to AD and also to AC, it is $\perp$ to the plane DAC, and $\therefore$ to every st. line in that plane, and $\therefore$ to AE and DE.

But, by construction, AE is $\perp$ to CD, and $\therefore$ DE is $\perp$ to AE and also to AB, and $\therefore$ to the plane passing through AE, AB, and $\therefore$ to every st. line in this plane, and $\therefore$ to EB.

13. Let A be a pt. in the wall XOO', and B a pt. in the wall YOO'. The shortest distance from A to B measured along the planes is ACB, where AC, CB make equal angles with $O'O''$: for if we imagine the plane YOB to be turned about the intersection OO' till it coincides with the plane XOA, then ACB will be a st. line, and $\therefore$ the shortest distance between A and B. To find the pt. C, draw AO'' in the plane XOO', $\perp$ to OO', and BO' in the plane YOO' $\perp$ to OO'. Divide $O'O''$ in C so that $O''C : O'C = AO'' : BO'$. Then since Δs $AO''C$, $BO'C$ have $\angle$s at O'', O' rt. $\angle$s, and the sides about them proportional,

$$\therefore \angle ACO'' = \angle BCO';$$

$\therefore$ C the required pt. is found.

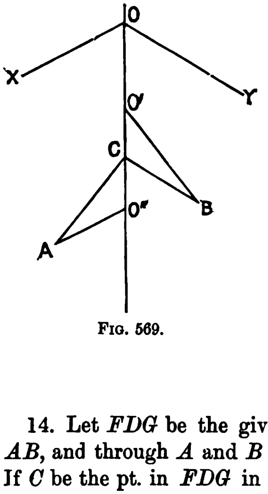


FIG. 569.

14. Let FDG be the given plane, A and B the given pts. Join AB, and through A and B draw the $\perp$ s AD, BE to the plane FDG. If C be the pt. in FDG in which the lines from A and B intersect when the distance is the shortest possible, it must lie in the plane through $AB \perp$ to FDG, and $\therefore$ in the line DE. For if not, suppose C' to be the point; and draw $C'C$ $\perp$ to DE in plane FDG, and join AC', BC'. Then we have the right-angled Δs ACC', BCC', and the sum of the hypotenuses AC', BC' must be greater than the sum of AC, BC; and $\therefore$ C' cannot be the point, and $\therefore$ the point must be in DE.

Again, if we produce AD to A', so that $A'D=AD$, it is plain that A' is at the same distance from any pt. C in the plane FDG as A is, and $\therefore$ if we join BA', BA' is the shortest distance between B and A', and if BA' meets the plane FGD in C, BC, CA' together equal BC, CA together, and C is the required pt. But BCA' is a straight line, and
$\therefore \angle BCE = \angle DCA' = \angle ACD$.

Hence the sum of the two st. lines is the least possible when they are drawn to the intersection of the plane through A and $B \perp$ to the given plane, and make equal angles with it.

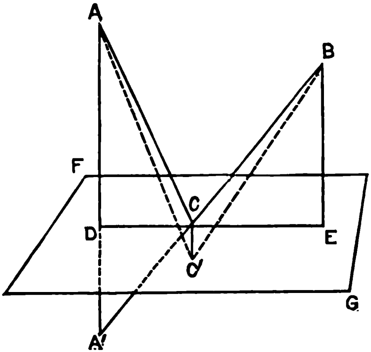


FIG. 570.

15. The two planes ABC, abc, are cut first by a plane $ABba$, in AB, ba, and next by a plane $ACca$, in AC, ca.

By XI. 16, AB is parallel to ab,
and AC is parallel to ac,
and since AB, AC, two st. lines that meet, are respectively parallel to ab, ac, two other st. lines that meet,
$\therefore$ by XI. 10, $\angle BAC = \angle bac$.

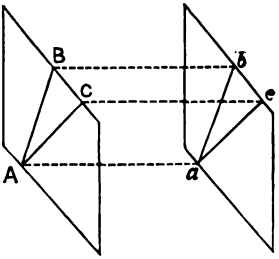


FIG. 571.

16. In the $\triangle$s BDA, BCA the two sides BD, DA are by hypothesis equal respectively to AC, CB, and the base AB is common,

$\therefore \angle BDA = \angle ACB$.

Similarly,
$\angle ADC = \angle ABC$, and $\angle CDB = \angle CAB$;
$\therefore \angle BDA + \angle ADC + \angle CDB = \angle ACB + \angle ABC + \angle CAB$,
$= 2$ rt. $\angle$s.

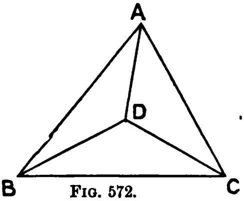


FIG. 572.

17. Let OA, OB, OC be the three st. lines.

Take $OA = OB = OC$, and join AB, BC, CA, which lie in the plane ABC.

In the plane ABC find D, the centre of the circle circumscribing the △ ABC, and join AD, BD, CD, OD.

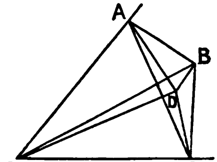


FIG. 573.

Then shall OD be the line required.

For in △s AOD, BOD,

∵ $AO = BO$, and OD is common, and $AD = BD$ (each being a radius of the ⊙ described about △ ABC),

∴ ∠ AOD = ∠ BOD.

Similarly, ∠ AOD = ∠ COD,

∴ the line drawn from O to the centre of the ⊙ described about △ ABC is the line required.

18. Let $ABCD$ be a triangular pyramid standing on the equilateral base BCD, and having the angles at A rt. angles.

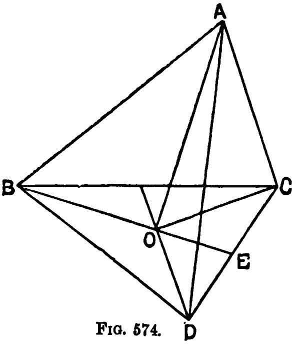


FIG. 574.

In △s ADB, ADC,

∵ $DB = DC$, and AD is common, and ∠ DAB = ∠ DAC,

∴ $AB = AC$.

Similarly, $AB = AD$.

Then if O be the point of intersection of ⊥s from the angular points of the △ BDC on the opposite sides,

∵ $OD = OB$, and OA is common, and $AD = AB$,

∴ ∠ AOD = ∠ AOB;

and similarly, ∠ AOD = ∠ AOC.

Hence AO is ⊥ to plane BDC.

Then sq. on AO = sq. on AD − sq. on DO,

= sq. on AD − $\frac{1}{3}$ sq. on DB,

= sq on AD − $\frac{2}{3}$ sq. on AD,

= $\frac{1}{3}$ sq. on AD.

19. The three plane angles BOA, COA, COB form a solid angle at O, COB being a rt. ∠, and COA the supplement of BOA, which we will take as acute. ABC is a plane cutting the edges OA, OB, OC in A, B, C, so that $OB = OC$.

Then in △ AOB

sq. on AB = sq. on OA + sq. on OB − 2 rect. contained by OB and the section of OB between O and foot of ⊥ from A on OB;

sq. on AC = sq. on OA + sq. on OC + 2 rect. contained by OC and the section of OC between O and foot of ⊥ from A on OC;

sq. on BC = sq. on BO + sq. on OC;

∴ sq. on AB + sq. on AC + sq. on BC = 2 (sq. on OA + sq. on OB + sq. on OC).

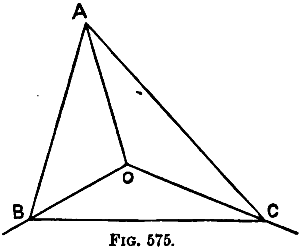


FIG. 575.

20. In the solid ∠ formed by OP, OA, OB ∠ POA + ∠ POB is greater than ∠ AOB; and in the solid ∠ formed by OP, OB, OC, ∠ POB + ∠ POC is greater than ∠ BOC; and in the solid ∠ formed by OP, OC, OA, ∠ POC + ∠ POA is greater than ∠ COA;

∴ ∠ POA + ∠ POB + ∠ POC is greater than $\frac{1}{2}$ (∠ AOB + ∠ BOC + ∠ COA).

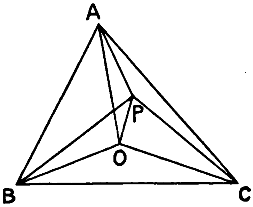


FIG. 576.

Page 342.

Senate-House Riders on Books VI. XI. and XII.

1849. VI. 4. Let AOD be a chord passing through the fixed point O in a circle. Draw any other chord BOC. Join AB, CD. Then △s AOB, COD are similar;

∴ $AO : OB = CO : OD$;

and ∴ rect. AO, OD = rect. OB, OC;

that is, the rectangle under the segments of any chord passing through O is constant.

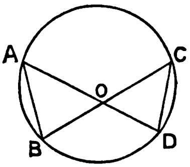


FIG. 577.

XI. 11. This has been proved in Ex. 7 on page 334.

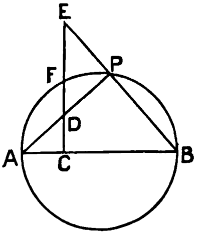


FIG. 578.

1850. VI, 10. Since the $\angle$s at P are rt. $\angle$s,
$\therefore \triangle$s ADC, EDP are similar;
and $\therefore \triangle$s ECB, ACD are similar;
$\therefore AC : CD = EC : CB$;
$\therefore$ rect. CD, $EC =$ rect. AC, CB,
$=$ sq. on CF;
$\therefore CE : CF = CF : CD$.

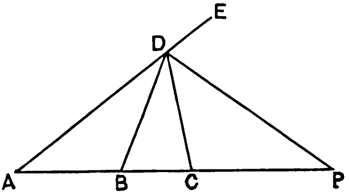


FIG. 579.

1851. VI. 3. Produce AD to E, and draw DP bisecting the $\angle\ CDE$. Then

$$\angle PDB = \tfrac{1}{2}(\angle ADC + \angle CDE).$$

$\therefore \angle PDB =$ a rt. $\angle$,

$\therefore$ a ⊙ described on BP as diameter passes through D.

(See also Ex. 83 on p. 302.)

XI. 8. Through E draw the plane $CAD \perp$ to AB, intersecting the planes in AC, AD, and in the plane $ACED$ draw EC, ED $\perp$s to AC, AD.

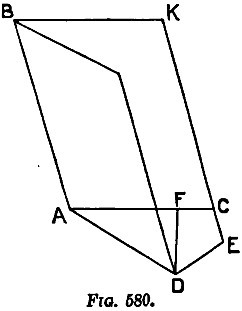


FIG. 580.

Through C draw, in the plane BAC, $CK \parallel$ to AB.

Then ECK is a rt. $\angle$, and also ACE is a rt. $\angle$.

$\therefore EC$ is $\perp$ to plane BAC.

Similarly, ED is $\perp$ to plane BAD.

Draw $DF \perp$ to AC in plane CAD.

Then by drawing from F in plane BAC a line $\parallel$ to AB, it may be shown that DF is $\perp$ to plane BAC.

But F is a point in the line AC, and $\therefore CF$ produced is $\perp$ to AB.

1852. VI. 2. This has been proved in page 294, Ex. 12.

XI. 11. O is the point in which the perpendiculars from the angular points of BCD on the opposite sides intersect.

(See Ex. 18, p. 335.)

Then sq. on AD = sq. on AO + sq. on DO,
= sq. on $AO + \frac{1}{3}$ sq. on BD;
∴ 3 sq. on AD = 3 sq. on AO + sq. on AD;
∴ 2 sq. on AD = 3 sq. on AO;
∴ 2 sq. on AB = 3 sq. on AO.

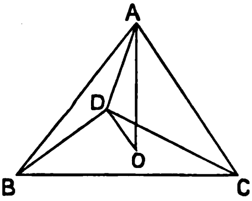


FIG. 581.

1853. VI. 6. Describe a ⊙ round ABC; then CD cuts arc AB in E' its middle point, and if the base AB and the vertical angle at C be given, E' is a fixed point.

From CA measure $CF = CB$; join FD, FE', $E'B$.

Then ∵ $CF = CB$, CD is common, and
∠ FCD = ∠ BCD,
∴ $DF = DB$, and similarly $FE' = BE'$.
Hence ∠ $DE'F$ = ∠ $DE'B$,
= ∠ DAF;
∴ a ⊙ can be described about $FAE'D$;
∴ rect. CD, CE' = rect. CA, CF,
= rect. CA, CB,
= rect. CD, CE (by the question);
∴ E coincides with E', a fixed point.

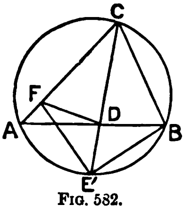


FIG. 582.

XI. 21. In the diagram A', B, C, A are supposed to be in the plane of the paper, and D above it.

Then since BA, CA are respectively ⊥ to the planes $BA'D$, $CA'D$, it follows that $A'D$, the intersection of the planes, is ⊥ to BA and CA, and therefore is ⊥ to the plane containing them, that is $BA'C$.

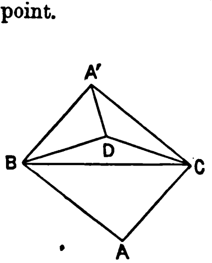


FIG. 583.

∴ $DA'B$ and $DA'C$ are rt. ∠s, and ∠ $BA'C$ is the supplement of ∠ BAC,
∴ ∠s at A' together with ∠ BAC make four rt. ∠s.

1854. VI. 16. Let O, O' be the respective centres; join CO, $C'O'$, BE, $B'E$.

Then since BE is || to $O'C'$;

$$\therefore AB : BC' = AE : EO';$$

$$\text{or } AB : BC' = AE : \tfrac{1}{2} A'E.$$

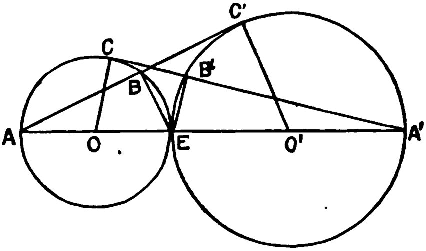


FIG. 584.

Similarly, $A'B' : B'C = A'E : \frac{1}{2} AE$.

Compounding, $AB, A'B' : BC', B'C = AE, A'E : \frac{1}{4} . A'E, AE,$

$$= 4 : 1;$$

$$\therefore \text{rect. } AB, A'B' = 4 \text{ rect. } BC', B'C.$$

XI. 20. The inner triangle need not have its sides parallel to those of the outer. Let abc be the inner and ABC the outer. Let ab, bc, ca be produced to meet the sides of ABC in γ, α, β. Join bC, and let O be the point not in the plane of the triangles; and let any line, such as ac, stand for the angle subtended by it at O.

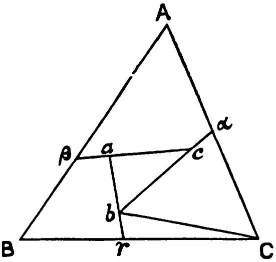


FIG. 585.

Then $b\gamma + \gamma C$ is greater than bC; (XI. 20.)

$\therefore b\gamma + \gamma C + Ca$ is greater than $bC + Ca$;

$\therefore b\gamma + \gamma C + Ca$ is greater than ba;

$\therefore b\gamma + \gamma C + Ca$ is greater than $bc + ca$;

so also, $ca + aA + A\beta$ is greater than $ca + a\beta$;

and, $a\beta + \beta B + B\gamma$ is greater than $ab + b\gamma$;

$\therefore$, adding, and removing equal lines from each side,

$\gamma C + Ca + aA + A\beta + \beta B + B\gamma$ is greater than $bc + ca + ab$;

that is, $CA + AB + BC$ is greater than $bc + ca + ab$,

$\therefore$ the particular case is also proved.

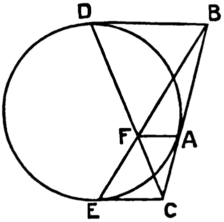


FIG. 586.

1855. VI. 2. Since BD and EC are parallel,

∴ △s BFD, EFC are similar,

∴ $FB : FE = BD : EC$.

Now $BD = BA$, and $EC = CA$,

∴ $FB : FE = BA : CA$,

∴ AF is parallel to BD and EC.

XI. 16. The planes aB, cD are evidently parallel;

∴, since they are cut by the plane ad, ab and cd are parallel.

From a draw aE parallel to AB; aE will evidently lie in the plane aB, and will intersect Bb in some point E, so that $aEBA$ shall be a parallelogram, and ∴ $aE = AB$.

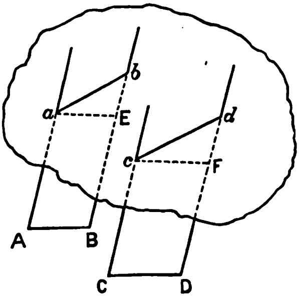


FIG. 587.

Similarly, draw $cF \parallel$ to CD to intersect Dd in F, then, as before, $cF = CD$.

Now, since ab is $\parallel$ to cd, and aE is $\parallel$ to AB, which is $\parallel$ to CD, which is $\parallel$ to cF, ∴ aE is $\parallel$ to cF.

Hence $\angle baE = \angle dcF$.

Similarly, $\angle abE = \angle cdF$,

∴ △s abE, cdF are similar;

∴ $ba : aE = cd : cF$;

and ∴ $ba : AB = cd : CD$. (I. 34.)

1856. XI. 11. Let $ABCD$ be the regular tetrahedron.

Bisect CD in E. Join AE, BE, and take F in BE such that $BF = 2\,FE$. Join AF.

Now F is the pt. where the ⊥ from A on BCD meets BCD (see p. 335, Ex. 18). But all such ⊥ s in a regular tetrahedron are equal; and hence if BG be the ⊥ from B on ACD, $AF = BG$.

Draw $FH \perp$ to AE. Then by similar △s BGE, FHE,

$BG : FH = BE : FE$;

$= 3 : 1$.

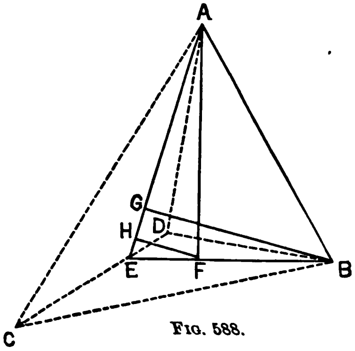


FIG. 588.

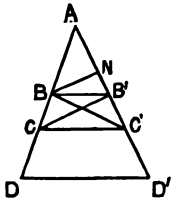


FIG. 589.

1857. VI. 19. The triangle $AC'B$ always varies as its base and height jointly. Draw $BN \perp$ to AB', then $AC'B$ varies as the rectangle AC', BN.

Suppose $B_{,}B'_{,}$, $C_{,}C'_{,}$ to be new positions of BB' and CC': then ($N_{,}$ being the new position of N) the triangle $AC'_{,}B_{,}$ would vary as the rectangle $AC'_{,}$, $B_{,}N_{,}$.

But the $\triangle$s BNB', $B_{,}N_{,}B'_{,}$ are similar;

$\therefore \triangle AC'B : \triangle AC'_{,}B_{,} =$ rect. AC', BN : rect. $AC'_{,}, B_{,}N_{,}$

$=$ rect. $AC', B'B$: rect. $AC'_{,}, B'_{,}B_{,}$

$=$ rect. $CC', B'B$: rect. $C_{,}C'_{,}, B'_{,}B_{,}$

that is, $\triangle ACB$ varies as rect. CC', BB'.

Similarly, the proposition may be shown to be true for the $\triangle AB'C$.

XI. 16. Let $OABC$ be the pyramid, ABC the equilateral base.

Bisect AB in F, join CF, OF. CF is evidently $\perp$ to AB, and $AB \perp$ to plane OCF, and $\therefore$ to OF. But F is the middle pt. of AB, and $\therefore$ $\triangle$s OFB, OFA are equal in every respect;

$\therefore OB = OA$; and, by similar reasoning, $OB = OC$.

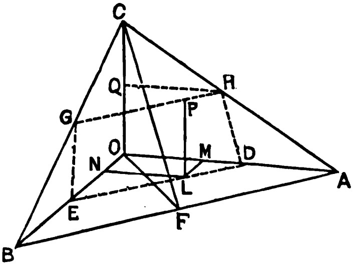


FIG. 590.

Take any pt. P in the plane ABC; draw $PL \perp$ to OAB, through PL draw the plane $EPD \parallel$ to AB and cutting OAB in ED, and CAB in GH (where evidently, by construction, both ED and GH will be $\parallel$ to AB). The $\perp$s from P on the planes OAC, OBC are evidently respectively equal to LM and LN.

For any position of P along GH, the $\perp$ on the plane OAB, will always be equal to PL. Hence for positions of P along GH, the sum of the $\perp$s will always be the same, if the sum of LM and LN be the same.

In Fig. 591, let L', M', N' be new positions of L, M, N; $L'M'$, LN intersecting in K, then evidently $LM+LN = L'M' + L'N'$, if $L'K=LK$, which is true, because $L'KL$ is a △ similar to BOA, and ∴ isosceles. Hence for any pt. P in GH the sum of the ⊥s is equal to $PL+LM+LN$, that is, to $PL+DO$, or $HD+DO$, or $HD+HQ$ (HQ being ∥ to DO).

Exactly as before, $HD+HQ$ is of constant value and equal to AO.

But P was taken anywhere in ABC, and the sum of its ⊥s has been found to be equal to AO, and ∴ the sum of the ⊥s from any point in the △ ABC (and *within* it) is constant and equal to AO (or BO or CO).

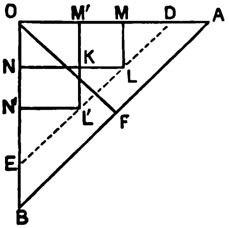


FIG. 591.

The same is true if P be in the plane ABC and *without* the △ ABC, provided that the ⊥s be subtracted when they fall on the side of the △s OAB, OBC, OCA opposite to that on which they fell when P is *within* the △ ABC.

1858. VI. 15. Let ABC be the △, and BC its base.

Take P in AB such that

sq. on $AP=$rect. AB, BP. (II. 11.)

Draw PD ∥ to BC, to meet AC in D, and join PC.

Then shall P be the point required.

For since sq. on $AP=$rect. AB, BP,

$\therefore AP:PB=AB:AP$;

and ∴, since △s APD, ABC are similar,

$AP:PB=BC:PD$;

and $\angle APD = \angle PBC$;

∴ (by VI. 15) △ APD = △ PBC.

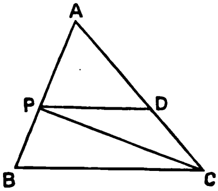


FIG. 592.

XI. 11. Cut the given planes by a plane ⊥ to their line of intersection. Let the given planes cut this plane in the lines XOX', YOY'. Each required locus will reduce to a point where it cuts the above plane of construction (the plane of the paper). P, Q, R, S are these points.

Bisect ∠ YOX by OP, and from O draw $ON \perp$ to YO, and equal to the given line in length; draw $NP \parallel$ to OY to cut OP in P; and draw PM, PL ⊥s to OY, OX. The △s POM, POL are evidently equal, and ∴ $PM = PL =$ given line.

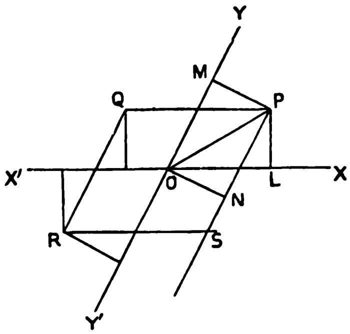


FIG. 593.

Construct similarly for R. Through P and R draw PQ, $RS \parallel$ to OX, and PS, $RQ \parallel$ to OY, intersecting in Q and S.

Then P, Q, R, S are evidently the required points.

Moreover, PQ, QR, RS, SP are lines of intersection of the four planes (of the second part of the rider) with the plane of the paper. Perpendiculars from O on these lines (ON is one of them) are evidently all equal to the given line.

1859. VI. 31. By VI. 18 this rider is self-evident: it being only necessary to notice that if two or more sides of the given polygon are equal, an equal number of the described polygons will be equal, so that there will be only as many polygons of *different* sizes as there are sides of *different* sizes in the original polygon.

XI. 20. Let the four lines be cut by a plane in the points A, B, C, X, the lines being OA, OB, OC, OX. Join AB, BC, CA, AX, BX, CX; and let the ∠s subtended at O by AB, BC, CA, AX, BX, CX be represented respectively by c, a, b, x, y, z.

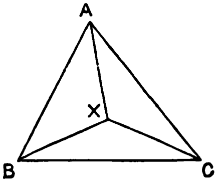


FIG. 594.

Then (see solution of rider for 1869, XI. 20)

$c + b$ is greater than $y + z$,

$a + c$ is greater than $x + z$,

$b + a$ is greater than $y + x$;

∴ $a + b + c$ is greater than $x + y + z$.

Again, in the pyramid $OABX$, by XI. 20,

$x + y$ is greater than c,

so also, $y + z$ is greater than a,

and $z + x$ is greater than b;

∴ $2(x + y + z)$ is greater than $a + b + c$;

∴ $x + y + z$ is greater than $\frac{1}{2}(a + b + c)$.

1860. VI. A.—I. The proposition says that when $\angle DCB$ is bisected by CX, $AX : BX = AC : BC$.

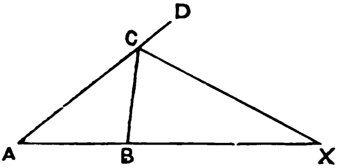


Fig. 595.

Now in the mind suppose CX always to bisect $\angle DCB$, while the triangle changes itself gradually to suit X moving away from A and B indefinitely, while C's position is always the same. Then evidently CX will come to fulfil Euclid's definition of parallelism with regard to AB. When this is the case,

$$\angle DCX = \angle CAX, \text{ and } \angle DCX = \angle BCX = \angle CBA\,;$$

$$\therefore \angle CAX = \angle CBA, \text{ or } CA = CB.$$

Hence when $CA = CB$ the external bisector CX is || to base AB.

II. Taking an evident construction, since CG bisects $\angle ACB$,

$$\therefore AG : GB = AC : CB\,;$$

$$\text{or, } AG : AC = GB : CB.$$

Let the bisector AF of the external $\angle BAD$ meet CG produced in F; then

$$FG : FC = AG : AC,$$
$$= GB : CB\,;$$

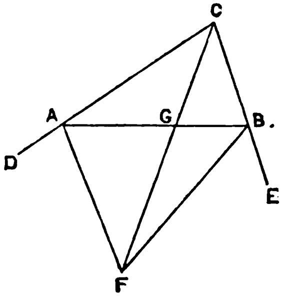


Fig. 596.

$\therefore$ joining FB, by VI. A. FB is the bisector of external $\angle ABE$, that is, the external bisectors of A and B meet the internal bisector of C in the same point.

XI. 17. Let *ACE*, *GHK*, *BDF* be the three planes, of which *ACE* and *GHK* are parallel; *AGB*, *CHD*, *EKF* the three lines cut by these planes, so that $AG : CH : EK = GB : HD : KF$.

From *C* draw *CLMN* || to *AGB*, cutting the three planes in *C*, *L*, *N*; join *AC*, *GL*, *BN*; *LH*, *ND*.

Now *AL* is a parallelogram; and so also is *GN*, if the plane *BDF* is || to *GHK*, and in this case the rider is proved by XI. 17.

If this is not so, take *LM* (in *LN*) = *GB*, and join *BM*, *MD*. Then *GM* is a ▱;

$$\therefore AG : GB = CL : LM;$$
$$\therefore CH : HD = CL : LM;$$
$$\text{and } \therefore LH \text{ is } \parallel \text{ to } MD;$$

∴ *BDM* is a plane || to *GHK*, and different from the plane *BDF*.

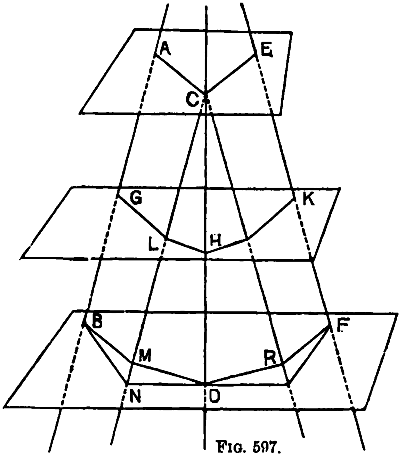


FIG. 597.

In a similar manner another plane *DRF* may be found passing through *D* and *F*, and || to *GHK*, and different from the plane *BND*.

But two planes passing through the same point *D*, and || to *GHK*, must be coincident, that is, the planes *BMD*, *DRF* must be the same plane, and this plane must evidently be *BDF*, for otherwise, two planes not coincident may be made to pass through three points not in the same straight line, which is impossible.

∴ *the plane BDF* coincides with both of the planes *BMD*, *DRF*, *and is || to GHK* and *ACE*.

1861. VI. 6. Let the diagonals intersect in O. Join EF, FG, GH, HE. In $\triangle AOD$, H and E are two of the feet of perpendiculars, and hence $\triangle$s OEH, OAD are similar,

and $\therefore \angle OHE = \angle ODA$.

Similarly, in the $\triangle$s OAB, OGH,

$$\angle OHG = \angle OBA;$$

$$\begin{aligned} \therefore \angle EHG &= \angle ODA + \angle OBA, \\ &= \angle ODA + \angle ODC, \\ &= \angle ADC. \end{aligned}$$

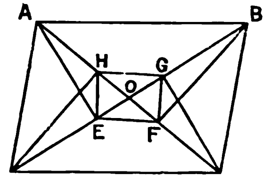


FIG. 598.

Again, since the pairs of $\triangle$s OEH, OAD; OHG, OBA, are similar,

$$\begin{aligned} \therefore EH : AD &= OH : DO, \\ &= OH : BO, \\ &= HG : BA; \\ \text{or, } EH : HG &= AD : BA, \\ &= AD : DC. \end{aligned}$$

$\therefore$ the quadrilateral $EFGH$ is similar to $ABCD$, and $\therefore$ is also a parallelogram.

XI. 12. Let $ABCD$ be the tetrahedron, and E the middle pt. of CD; F a pt. in EB such that $FB = 2\,EF$. Join AE, AF.

Let one edge of the tetrahedron be $\sqrt{3}$ in length.

Then, by I. 47, $EB = \frac{3}{2}$, and if G be the middle pt. of AB, $BG = \frac{\sqrt{3}}{2}$.

Now the perpendicular distance between the two lines CD, AB is the shortest distance between them, and this distance is EG.

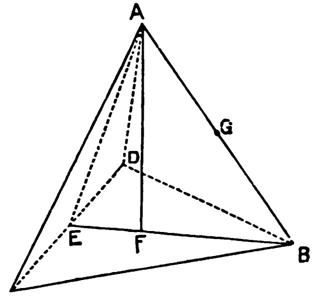


FIG. 599.

Now, by I. 47, sq. on EG + sq. on GB = sq. on EB;

$$\therefore \text{sq. on } EG + \frac{3}{4} = \frac{9}{4}, \text{ and } \therefore EG = \frac{\sqrt{6}}{2}.$$

The diagonal of a square described on an edge = the edge × $\sqrt{2} = \sqrt{6}$;

$$\therefore \text{half of this diagonal} = \frac{\sqrt{6}}{2} = EG.$$

1862. VI. 1. By a well-known theorem in Modern Geometry, if in the △ ABC, AD, BE, CF meet in O.

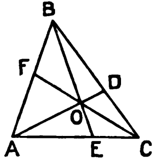


FIG. 600.

$AE \cdot BF \cdot CD = AF \cdot BD \cdot CE$.

Then (1) when $AF : FB = CD : DB$,

we have $AE : CE = AF \cdot DB : BF \cdot CD$,

$= CD \cdot BD : DB \cdot CD$,

$= 1 : 1$.

And (2) when $AF : FB = BD : DC$,

we have $AE : CE = AF \cdot BD : BF \cdot CD$,

$=$ sq. on BD : sq. on CD.

XI. 21. This rider has been already proved in page 335, Ex. 16.

1863. VI. 4. Let ABC be the △, AD the external bisector of A, BE and CF the internal bisectors of B and C, meeting the sides in D, E, and F.

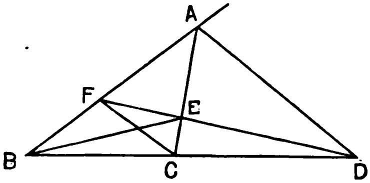


FIG. 601.

Then from the proposition in Modern Geometry, that if three st. lines AD, BE, CF cut the sides of the △ ABC in D, E, F, so that DEF is a st. line, then $AE \cdot BF \cdot CD = AF \cdot BD \cdot CE$;

and since, by VI. 3,

$$CE : EA = CB : BA,$$

$$\text{and } BF : FA = BC : CA,$$

and, by VI. A.,

$$BD : DC = BA : AC;$$

∴ the equation $AE \cdot BF \cdot CD = AF \cdot BD \cdot CE$ becomes

$BA \cdot BC \cdot AC = CA \cdot BA \cdot CB$, an identity,

∴ DEF is a st. line.

XI. 17. Let $ABCD$ be the tetrahedron, E, F the middle pts. of AD, BC respectively. Join DF, EC, EF, EB.

Then sq. on DC + sq. on DB = 2 (sq. on DF + sq. on FC).

By II. 13, Ex.

Also, sq. on DC + sq. on DB = sq. on DE + sq. on EC − 2 rect. $EC.En$
+ sq. on DE + sq. on EB − 2 rect. EB, Em,
= 2 sq. on DE + sq. on EC + sq. on EB − 2 rect. $EC.En$ + 2 rect. EB, Em,
when n, m are the feet of $\perp$s from D on EC and EB.

But $DC = AB$, and $DB = AC$.

∴ sq. on DC + sq. on DB = sq. on AB + sq. on AC,
= 2 sq. on EA + sq. on EB + sq. on EC − 2 rect. EB, Em' = 2 rect. EC, En',
where m', n' are the feet of $\perp$s from A on BE, CE produced ; also
$Em = Em'$, and $En = En'$.

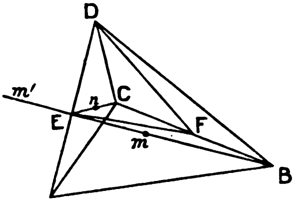


Fig. 602.

∴ adding these two values of (sq. on DC + sq. on DB),
2 (sq. on DC + sq. on DB)
= 2 (sq. on DE + sq. on EA + sq. on EC + sq. on EB),
or, sq. on DC + sq. on DB
= 2 sq. on DE + sq. on EC + sq. on EB (for $EA = DE$) ;
∴ 2 sq. on DF + 2 sq. on FC = 2 sq. on DE + sq. on EC + sq. on EB,
= 2 (sq. on DE + sq. on EF + sq. on FC) ;
∴ sq. on DF = sq. on DE + sq. on EF.

Now DEF is a plane, and ∴, by I. 48, $\angle DEF$ is a rt. angle.

1864. VI. 23. The ▱s AC, BF are to one another in the ratio compounded of the two ratios $AB : BD$ and $CB : BE$, that is, the ratio $AB.BC : DB.BE$; it is required to show that $\angle ABC = \angle DBE$.

If not, let BF' be the ▱ which is to the ▱ AC in the ratio $DB.BE : AB.BC$, where ∴ $D'B = DB$ and $\angle D'BE$ is not = $\angle CBA$, that is, $\angle DBE$.

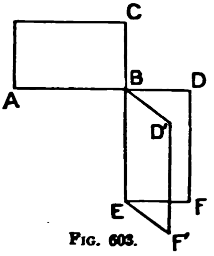


Fig. 603.

And now construct the ▱ DE equiangular with AC, and ∴ such that

$AC : DE = AB.BC : DB.BE$.

But, by hypothesis, $AC : D'E = AB\,.\,BC : DB\,.\,BE$;

∴ ▱ $D'E =$ ▱ DE, that is, on the same base, and *not* between the same parallels (though on the same side of the base), two ▱s have been constructed which are equal, which is impossible.

(I. 35, Cor.)

Hence ▱s AC, BF must be equiangular.

XI. 12. Let ABC be the given equilateral △.

Aa, Bb, Cc are the ⊥s on opposite sides, meeting in O. At O raise a ⊥ OD of length such that its square is twice the square on AO. Join AD, BD, CD.

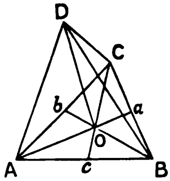


Fig. 604.

Then $ABCD$ is the required tetrahedron.

The solid is evidently symmetrical as regards AD, BD, and CD, and it is only necessary to prove that $AD = AB$. Now sq. on AD

$= $ sq. on $AO +$ sq. on OD,

$= $ sq. on $AO +$ sq. on $AO +$ sq. on AO,

$= $ sq. on $AO +$ sq. on $OB + 2$ rect. AO, Oa,

$= $ sq. on AB.

∴ all the six sides of the tetrahedron are equal, and it is regular.

1865. VI. 19. Since $\triangle FAE : \triangle CAE = FA : AC$,

and $\triangle FDE : \triangle BDE = FD : DB$,

$\therefore \triangle FAE \times \triangle BDE : \triangle FDE \times \triangle CAE = FA \times DB : FD \times AC$;

$\therefore \triangle FAE \times$ sq. on $DE : \triangle FDE \times$ sq. on $AE = FA \times DB : FD \times AC$;

i.e. (since △s BFA, CFD are similar) $= AB \times DB : CD \times AC$;

i.e. (since △s BED, CEA are similar) $= AB \times DE : CD \times AE$;

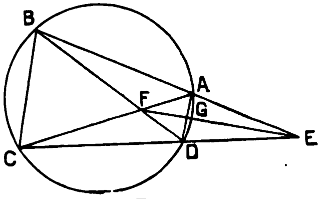


Fig. 605.

$\therefore \triangle FAE \times DE : \triangle FDE \times AE = AB : CD$;

or, $\triangle FAE : \triangle FDE = EA \times AB : CD \times DE$.

But $AG : GD = \triangle FAE : \triangle FDE$;

$\therefore AG : GD = EA \times AB : CD \times DE$.

XI. 16. Through AB draw the plane $AbNB \perp$ to CD.

Through AC draw the plane $AcNC \perp$ to BD.

Their line of intersection AN is $\therefore \perp$ to CD and BD, *i.e.* AN is $\perp$ to the plane BCD.

Any plane through AN is $\therefore \perp$ to the plane BCD, and $\therefore$ the plane DAN is $\perp$ to BCD.

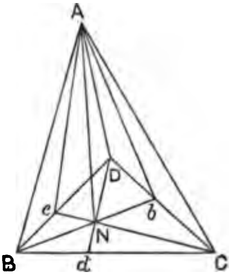


FIG. 606.

Join DN and produce it to meet BC in d.

Now Cc, Bb are the $\perp$s from C and B on opposite sides, and $\therefore Dd$ is the third $\perp$.

Hence the plane ANd has two lines in it, viz., AN and Nd, both $\perp$ to BC, and $\therefore$ it cuts BC perpendicularly;

$\therefore BC$ is $\perp$ to every line in the plane ANd;

$\therefore BC$ is $\perp$ to AD.

1866. VI. 4. Draw $BO \perp$ to BC, meeting CF produced in O. Then $OB = 2DE$.

Now AFD, BFO are similar $\triangle$s.

$$\therefore AF : FB = AD : OB,$$
$$= \tfrac{1}{2} AD : DE;$$
$$\therefore 2AF : FB = AD : DE.$$

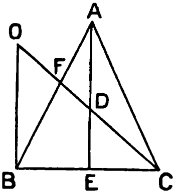


FIG. 607.

1867. VI. A. Let ABC be the $\triangle$, right-angled at C, Aa, Aa', Bb, Bb', the four bisectors, meeting the sides in a, a', b, b'.

Then $Ca : aB = CA : AB$; (VI. 3.)

$$\therefore Ca : CB = CA : CA + AB.$$

Also, $Ca' : a'B = CA : AB$; (VI. A.)

$$\therefore Ca' : CB = CA : AB - CA.$$

Hence $Ca + Ca' = aa' = CA \,.\, CB \,.\left(\dfrac{1}{CA+AB} + \dfrac{1}{AB-CA}\right)$;

$$\therefore aa' = \frac{2AB \,.\, BC \,.\, CA}{AB^2 - CA^2} = \frac{2AB \,.\, BC \,.\, CA}{BC^2} = 2AB \cdot \frac{AC}{BC}.$$

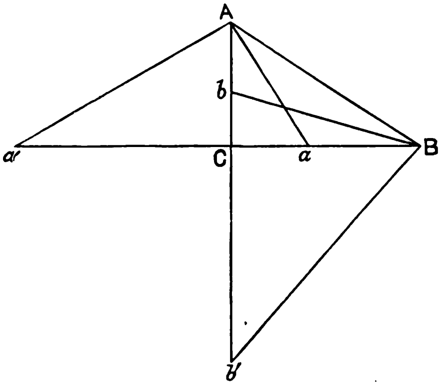


FIG. 608.

Similarly, $bb' = 2 \,.\, AB \cdot \dfrac{BC}{AC}$;

$$\therefore aa' \,.\, bb' = 4AB^2.$$

XI. 21. Looking down from above on the bases of the pyramids, we see two figures like those here shown.

$$\text{Now } 4AB = 8EF;$$

$$\therefore AB = 2EF.$$

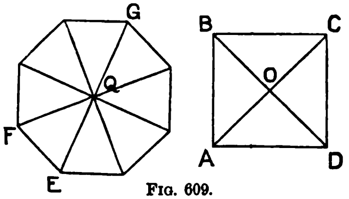


FIG. 609.

Also in the two *sides* (one from each pyramid) AOD, EQF, we have $AO = OD = EQ = QF$.

Bisect AD in H; join OH. Then construct the isosceles $\triangle\ OAK$, having $OK=OA$ and $AK=AH$. (I. 23.)

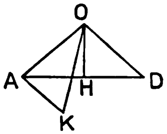


FIG. 610.

Then $\triangle\ AOK$ will be equal in every respect to $\triangle\ EQF$.

Now $\angle\ AOK$ is less than $\angle\ AOH$;

$\therefore\ 2\angle\ EQF$ is less than $\angle\ AOD$;

$\therefore$ sum of plane angles at Q is less than sum of plane angles at O.

1868. VI. 2. $\triangle BDE : \triangle BAE = BD : BA,$

$$= BE : BC,$$

$$= \triangle BAE : \triangle BCA;$$

$$\therefore\ \triangle BAE : \triangle BCA = BD : BA;$$

$\therefore$, compounding,

$$\triangle BDE : \triangle BCA = BD^2 : BA^2.$$

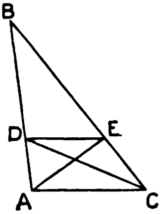


FIG. 611.

1868. XI. 11. This rider consists of two parts—(I.) To construct the "least angle" in question; (II.) To find when this "least angle" is greatest, and what it is.

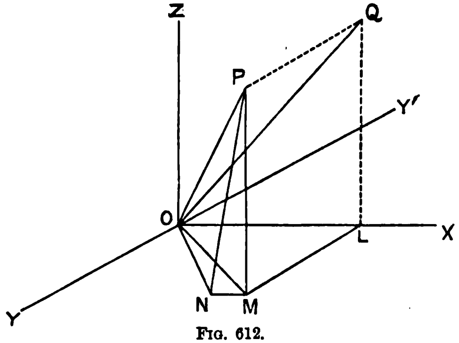


FIG. 612.

(I.) Let OP be the given line in the plane YPY', let YXY' be the other plane, intersecting the former in the line YOY'; it is required to draw from O the line in the plane XOY, which makes the least angle with OP of all such lines drawn in the plane XOY.

Draw $OZ \perp$ to XOY; let the plane ZOP cut the plane XOY in OM; from P, any pt. in OP, drop $PM \perp$ to XOY, and $\therefore$ in the plane ZOP, and $\therefore$ meeting OM in some pt. M; $\therefore \angle PMO$ is a right $\angle$. Draw any other line ON in the plane XOY; and draw $PN \perp$ to ON; join MN.

Since PM is $\perp$ to XOY, $\therefore PM$ is $\perp$ to MN, and $\therefore PN$ is greater than PM.

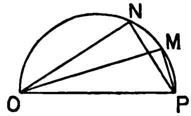


FIG. 613.

Now PNO, PMO are two right-angled $\triangle$s having the same hypotenuse OP, and may $\therefore$ be inscribed in the same semicircle of diameter OP, and hence the angle opposite PN in $\triangle OPN$ is at once seen to be greater than the angle opposite PM in $\triangle OPM$;

$\therefore \angle POM$ is less than $\angle PON$.

$\therefore$ the least angle for a given position of OP is that which lies in the plane through $OP \perp$ to the plane XOY.

(II.) Draw PQ, $ML \parallel$ to YO, to meet in Q and L respectively the plane ZOX, which is $\perp$ to YO; join OQ, QL. Then QM is a rectangle, and $QL=PM$. Also PQO is a right angle, and $\therefore PO$ is greater than QO; and MLO is a right angle, and $\therefore MO$ is greater than OL; $\therefore$ in the $\triangle$s QOL, POM, the bases QL and PM are equal, but the sides QO, LO are respectively less than the sides PO, MO; $\therefore$ (as may easily be deduced from I. 20) $\angle POM$ is, for any position of PO passing through O, less than $\angle QOL$ which lies in the plane through $O \perp$ to YOY'. Now OP is in the plane YOP, and PQ is $\parallel$ to YO, $\therefore QO$ is in the plane YOP and $\perp$ to YO; also XO is in the plane XOY, and $\perp$ to YO; $\therefore \angle QOX$ is the angle between the two given planes, and is $\therefore$ seen to be the greatest of the above-mentioned "least angles."

1869. XI. 20. The rider may be proved in three similar steps.

I. Let $A'BC$ be a plane through BC, A' being between A and D.
Then $\angle ABA' + \angle ABC$ is greater than $\angle A'BC$, (XI. 20.)
and $\angle ACA' + \angle ACB$ is greater than $\angle A'CB$, (XI. 20.)
$\therefore \angle ABC + \angle ACB + \angle ABA' + \angle ACA'$ is greater than $\angle A'BC + \angle A'CB$.

Now add $\angle BAD + \angle CAD$ to both sides ; then, noting that

$\angle BAD + \angle ABA' = \angle BA'D$, and $\angle CAD + \angle ACA' = \angle CA'D$,

$\angle ABC + \angle ACB + \angle BA'D + \angle CA'D$ is greater than

$\angle A'BC + \angle A'CB + \angle BAD + \angle CAD$.

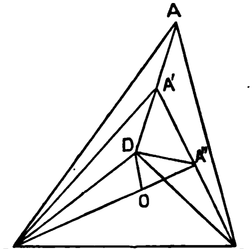


Fig. 614.

Add to the left-hand side $\angle A'BC + \angle A'CB + \angle BA'C = 2$ rt. $\angle$s, and to the right-hand side $\angle ABC + \angle ACB + \angle BAC = 2$ rt. $\angle$s.

Then cancelling like terms on both sides,

$\angle BA'C + \angle BA'D + \angle CA'D$ is greater than $\angle BAC + \angle BAD + \angle CAD$.

II. Let $A''BD$ be a plane through BD, A'' being between A and C ; then, as before,

$\angle BA''C + \angle BA''D + \angle CA''D$ is greater than $\angle BA'C + \angle BA'D + \angle CA'D$,

and $\therefore$ *a fortiori* greater than $\angle BAC + \angle BAD + \angle CAD$.

III. Let OCD be a plane through CD, O being between A'' and B ; and therefore, by the construction of I. and II., *within* the tetrahedron. Then we find, as before,

$\angle BOC + \angle BOD + \angle COD$ greater than $\angle BA''C + \angle BA''D + \angle CA''D$,

and $\therefore$ *a fortiori* greater than $\angle BAC + \angle BAD + \angle CAD$.

1870. VI. 15. Let OA, OB be the fixed lines ; AFB, $A'FB'$ two lines cutting off equal $\triangle$s AOB, $A'OB'$; E, E' their middle points ; CED, XFY, $C'E'D'$ parallels to the given direction.

Then it is required to prove that $CE \,.\, ED = C'E' \,.\, E'D'$.

By similar $\triangle$s

$$\left.\begin{array}{l} CE : XF = AE : AF \\ ED : FY = EB : FB \end{array}\right\} \text{ and } \left.\begin{array}{l} C'E' : XF = A'E' : A'F \\ E'D' : FY = E'B' : FB' \end{array}\right\} ;$$

$$\therefore CE \,.\, ED : XF \,.\, FY = AE \,.\, EB : AF \,.\, FB ;$$

$$\text{and } C'E' \,.\, E'D' : XF \,.\, FY = A'E' \,.\, E'B' : A'F \,.\, FB' ;$$

$$\therefore CE . ED : C'E' . E'D' = AE \,.\, EB \times A'F \,.\, FB' : A'E' \,.\, E'B' \times AF \,.\, FB.$$

$$= AE^2 \times A'F \,.\, FB' : A'E'^2 \,.\, AF \,.\, FB.$$

But, by construction, $\triangle AFA' = \triangle BFB'$, and $\therefore$, by VI. 15,

$$AF \,.\, FA' = BF \,.\, FB', \text{ and } \therefore \text{ the above proportion becomes}$$

$$CE \,.\, ED : C'E' \,.\, E'D' = AE^2 \,.\, FB'^2 : A'E'^2 \,.\, AF^2.$$

Hence the rider is true if

$$AE : A'E' = AF : FB', \text{ for then } CE \,.\, ED = C'E' \,.\, E'D'.$$

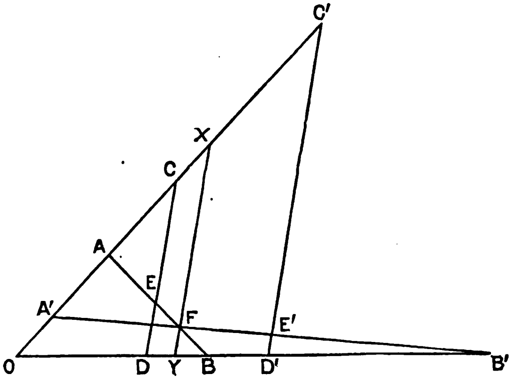


FIG. 615.

Now, if $AE : A'E' = AF : FB'$, then $AE : AF = A'E' : FB'$;

$$\therefore AE : EF = A'E' : E'F \text{ (for } E'B' = A'E');$$

$$\therefore AE - EF : AE + EF = A'E' - E'F : A'E' + E'F,$$

that is, $BF : AF = A'F : FB'$,

that is, $BF \,.\, FB' = AF \,.\, A'F$,

which is true, as above, by VI. 15;

$$\therefore CE \,.\, ED = C'E' \,.\, E'D'.$$

XI. 7. Let *ABCD* be the tetrahedron, such that any two opposite edges are $\perp$ to each other. From *A* drop a $\perp$ *AN* on the plane *BCD*, meeting it in *N*. Join *BN*, and produce it to cut *CD* in *B′*.

Now $\because$ *AN* is $\perp$ to the plane *BCD*, $\therefore$ it is $\perp$ to *CD*; but, by hypothesis, *AB* is $\perp$ to *CD*; and the two lines *AB*, *AN* intersect, and $\therefore$ lie in one plane. $\therefore$ the plane in which they lie, viz., *ABN*, is $\perp$ to *CD*, and $\therefore$ *CD* is $\perp$ to *BNB′* (XI. Def. 3.), that is, *BNB′*, *CNC′*, *DND′* are the $\perp$s on the sides of $\triangle BCD$; $\therefore$ *N* is the orthocentre of $\triangle BCD$.

Again, if a sphere pass through *A*, *B*, *C*, and *D*, the plane in which *the* $\triangle BCD$ lies will cut the sphere in a circle passing through *B*, *C*, *D*; *let L be the* centre of this circle, and Lb, Lc, Ld $\perp$s from *L* on *CD*,

DB, BC respectively; b, c, d being ∴ the middle pts. of these sides. Also, by a property of the orthocentre and centre of circumscribing ⊙ of a triangle, $BN = 2Lb$. In the plane ABN bisect BN in M, and draw $MK \parallel$ to NA; then, by VI. 2, K is the middle pt. of BA.

Again, the centre of the circumscribing sphere must lie in the line LO through L and $\perp$ to the plane BCD. Let O be the centre of the circumscribing sphere. Join KO. Let P be the orthocentre of the tetrahedron; join Pb. Then it is required to prove that KO is equal and parallel to Pb.

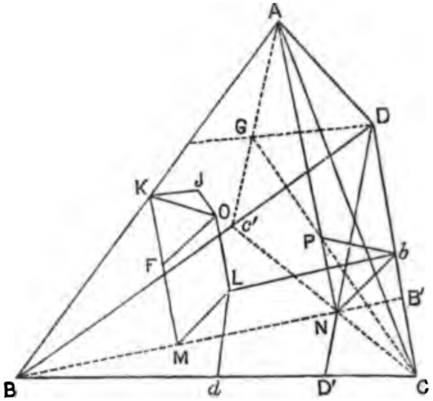


Fig. 616.

First, to prove that KO is $\parallel$ to Pb. The proof will depend on the following almost self-evident proposition:—"If two st. lines lie one in each of two parallel planes, and also one in each of other two parallel planes different from the former two, these two st. lines will be parallel." (See note at end of the proof.) Let G be the orthocentre and J the centre of the ⊙ circumscribing the △ ABD; then the planes KJO, CGD will be $\parallel$. For KJ is $\perp$ to AB, and JO is $\perp$ to the plane ABD, and ∴ $\perp$ to AB; ∴ AB is $\perp$ to the plane KJO.

And, as before, DG and CG are each $\perp$ to AB, and ∴ AB is $\perp$ to the plane CGD. Hence, by XI. 14, the planes KJO, CGD are parallel to each other, and KO lies in the plane KJO, and Pb lies in the plane CGD.

Again, ∵ MK, NA are both $\perp$ to the plane BCD, ∴, by XI. 6, they are ∥. Join ML, Nb, then LN is evidently a ▱, and ∴ ML is ∥ to Nb. Hence, since KM, ML are respectively ∥ to AN, Nb not in the same plane as KM, ML, the plane KML is ∥ to the plane ANb. (XI. 15.) But LO is $\perp$ to the plane BCD, as well as MK, and ∴ $KMLO$ is a plane, and is parallel to ANb. Now KO lies in the plane KML, and Pb lies in the plane ANb. Therefore, by the proposition enunciated above, KO and Pb are parallel.

In the plane $KMLO$ draw OF ∥ to ML, meeting KM in F.

Then FL is a ▱ ; and LN was shown to be one also ;

$$\therefore OF = ML = Nb.$$

Now KO is ∥ to Pb, OF is ∥ to Nb, and FK is ∥ to NP.

∴ △s KOF, PbN are similar, and they are also equal, for OF has been proved equal to Nb, and these are homologous sides, and ∴ $KO = Pb$.

Note.—To prove the assumed proposition. Let A, B, C, D be four planes, A ∥ to C, and B ∥ to D; and let A, B intersect in P, and C, D intersect in Q ; and B, C intersect in X. It is required to prove that P is ∥ to Q.

By XI. 16, P is ∥ to X ; and by the same proposition X is ∥ to Q, ∴, by XI. 9, P is ∥ to Q.

1871. VI. 2. By two well-known theorems of Modern Geometry,

(1.) Since AD, BE, CF meet in O,

$$AE \,.\, CD \,.\, BF = AF \,.\, BD \,.\, CE \,;$$

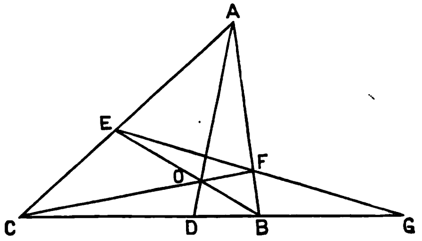


FIG. 617.

(2.) Since E, F, G are in the same line,

$$AE \,.\, CG \,.\, BF = AF \,.\, BG \,.\, CE \,;$$

$$\therefore AE \,.\, CD \,.\, BF \times AF \,.\, BG \,.\, CE = AF \,.\, BD \,.\, CE \times AE \,.\, CG \,.\, BF \,;$$

$$\therefore CD \,.\, BG = BD \,.\, CG \,;$$

$$\therefore BD : DC = BG : GC.$$

XI. 11. Let $ABCD$ be the tetrahedron; DN, CM two of the $\perp$s meeting in O. CM and DN will not meet unless the planes CDM, DCN coincide, that is, unless DM when produced meets CN when produced in one and the same point L in AB. Now the coincidence of these two planes, and the perpendicularity of AB to the coincident planes (for DN is $\perp$ to AB, and CM is also $\perp$ to AB), are co-extensive conditions, and $\therefore$ the $\perp$s CM, DN will not meet unless AB be $\perp$ to their common plane CLD, that is, unless $DB^2 - DA^2 = LB^2 - LA^2$. This condition might also be expressed under the form unless $CB^2 - CA^2 = LB^2 - LA^2$. Either of these conditions is necessary, and if the one holds the other must hold also.

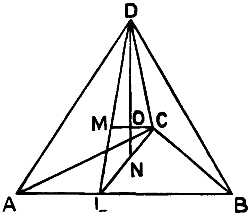


FIG. 618.

However, the pt. L is undetermined; we may then use these two conditions to get rid of reference to the pt. L; doing this, the conditions become the single condition $CB^2 - CA^2 = DB^2 - DA^2$, that is, $BC^2 + DA^2 = AC^2 + BD^2$. Hence the given condition is *necessary*.

It is also *sufficient*. For suppose a point L constructed for the $\perp$ DN, and a point L' for the $\perp$ CM; then from the condition would

$$CB^2 - CA^2 = DB^2 - DA^2,$$

while from these constructions would

$$CB^2 - CA^2 = L'B^2 - L'A^2, \text{ and } DB^2 - DA^2 = LB^2 - LA^2,$$

$$\therefore L'B^2 - L'A^2 = LB^2 - LA^2.$$

Now, by the axiom that the whole = sum of the parts,

$$\therefore L'B + L'A = LB + LA;$$

$$\therefore L'B - L'A = LB - LA;$$

$$\therefore 2\,L'B = 2\,LB;$$

$\therefore$ L and L' are the same point, $\therefore$ the plane $CDNL$ coincides with the plane $DCML'$, and $\therefore$ the $\perp$s CM, DN intersect.

COR. I. It may be noticed that when two $\perp$s CM, DN intersect, their plane is $\perp$ to the line of intersection of the two faces to which each is $\perp$; but CD is in this plane, and AB is the line of intersection of these two faces. Hence, when the $\perp$s intersect, the opposite edges are perpendicular.

COR. II. When the $\perp$s intersect, N is the orthocentre of the $\triangle ABC$, M that of the $\triangle ABD$.

1872. VI. 2. Let *PCP'* be the ⊙, *NQ* the st. line, *O* the pt.; *m* : *n* the given ratio.

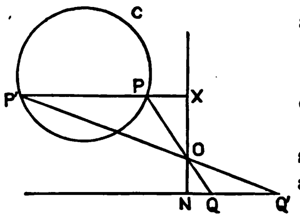


Fig. 619.

Draw *ON* ⊥ to *NQ*, cutting it in *N*, and produce *NO* to *X*, so that

$$ON : OX = m : n.$$

Through *X* draw *XPP'* || to *NQ*, to cut the ⊙ in *P* and *P'*.

Draw *POQ*, *P'OQ'* to cut the given st. line in *Q* and *Q'*; then *POQ*, *P'OQ'* are each cut by *O* in the ratio *n* : *m*.

For *XP* is || to *NQ*,

and ∴ $PO : OQ = XO : ON = n : m$.

Hence the construction is proved when *O* divides *PQ* *internally* in the given ratio.

The annexed figure shows *O* dividing *QP* *externally* in the given ratio. In each case there are seen to be two solutions except in the limiting positions, when the two coincide, *i.e.* when *XPP'* is a tangent to the ⊙.

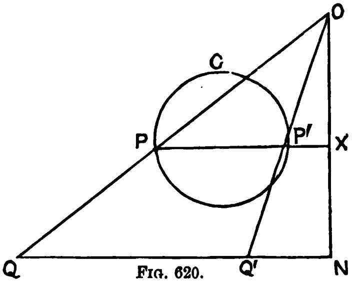


Fig. 620.

The next figure shows the possibility of four solutions, two internal and two external, when *O* lies between the two tangents to the ⊙, || to *NQ*.

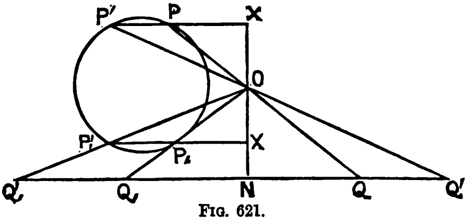


Fig. 621.

The same reasoning holds for the cases in which O lies *within* the ⊙.

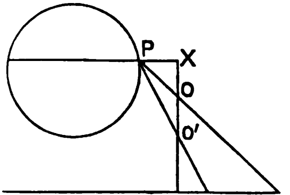


FIG. 622.

It may be noticed that for a given position of X and a given value of $m : n$, two positions of O may be found. Thus, in this fourth figure, take $ON : OX = m : n$ and two lines are found, of which POQ is one. Again, take $O'X : NO' = m : n$, and other two lines may be found, of which $PO'Q'$ is one.

For the limits of the ratio $m : n$ there will be three cases to consider.

Draw CA, DB tangents || to the given line, cutting NO in A and B respectively.

CASE I. When O lies between N and B, X must lie between B and A, and ∴ if we take $NO : OX = m : n$, we must have OX equal to or greater than OB, and less than or equal to OA, ∴ $NO : OX$ is equal to or less than $NO : OB$, and greater than or equal to $NO : OA$, that is, $m : n$ must not exceed the limits $NO : OA$ and $NO : OB$.

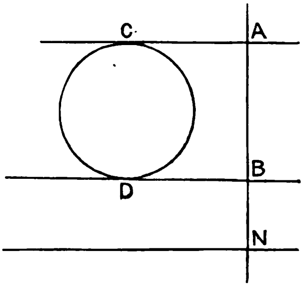


FIG. 623.

CASE II. When O lies between B and A, then as before X is also between B and A, and OX must be equal to or less than OB, $\left.\begin{matrix}\text{or}\\\text{and}\end{matrix}\right\}$ equal to or less than OA; that is, $m : n = NO : OX$ is equal to or greater than $NO : OB$, $\left.\begin{matrix}\text{or}\\\text{and}\end{matrix}\right\}$ equal to or greater than $NO : OA$;

that is, $m : n$ must not be less for external section than $NO : OB$, and not less for internal section than $NO : OA$.

Case III. When O lies further from N than A, then OX must be equal to or greater than OA, and less than or equal to OB; that is, $m : n = NO : OX$ must be equal to or less than $NO : OA$, and greater than or equal to $NO : OB$; that is, $m : n$ must not lie beyond the limits $NO : OA$ and $NO : OB$.

The point O might also lie on the other side of NQ from that on which the circle lies. This case is included in the former cases; and the limits of the ratio $m : n = NO : OX$ are $NO : OA$ and $NO : OB$.

XI. 20. **Part I.** $ABCD$ is the tetrahedron; $AB = CD$; $BD = CA$; and AD is common to $\triangle$s ABD, DCA;

$$\therefore \angle ABD = \angle ACD. \qquad (1.)$$

Similarly, $\angle ABC = \angle ADC$, and $\angle DBC = \angle DAC$. (2.)

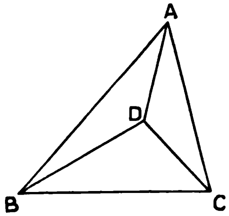


Fig. 624.

Now by XI. 20, any two of the $\angle$s at B are together greater than the third. Let $\angle$s ABD, DBC be together greater than $\angle CBA$, that is, by what has been proved, $\angle ACD + \angle DAC$ is greater than $\angle ADC$, *i.e.* the three $\angle$s ACD, CDA, and DAC are together greater than $2 \angle CDA$;

or, $2 \angle CDA$ is less than 2 rt. $\angle$s;

$\therefore \angle CDA$ is less that a rt. $\angle$.

The same holds for all the other plane $\angle$s.

Therefore each of the four triangles ABC, ACD, ADB, BCD is acute-angled.

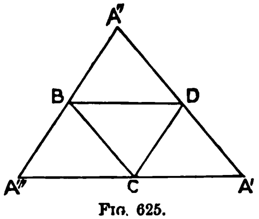


Fig. 625.

Part II. Let the four equal and similar acute-angled $\triangle$s be laid in one plane, as in the diagram, thus (the order of the letters indicating the equal sides), BDC, $A'CD$, DBA'', $CA'''B$.

A little consideration will show that this is the only possible relative position of the $\triangle$s. For evidently the side BD of BDA'' must lie along and coincide with BD of the $\triangle$ BDC. Suppose BD of BDA'' were reversed, then the side BA'' (on an attempt to form the tetrahedron) would have to coincide with DA', to which it is not equal. Therefore this is the only relative position. The only condition to which the three plane angles containing a solid angle are subject is (XI. 20) that *any* two are together greater than the third. Now if BDC be *not* acute-angled, let BDC be the angle greater than a right $\angle$; then $\angle A''DB$ is greater than a right $\angle$, and the $\angle$s BDC, DCB, *i.e.* $\angle$s BDC and CDA', are together less than a right $\angle$, and therefore less than $\angle A''DB$. Hence three such $\angle$s cannot enclose a solid $\angle$, and $\therefore$ the four equal and similar $\triangle$s must be acute-angled.

When this is the case, let the $\triangle$s $A''BD$, BDC, CDA' be so placed that DA'' coincides with DA', and since $DA''=DA'$, $\therefore$ A'' will coincide with A'; call this point A. Then we have BA, AC in one plane, and $\therefore$ BC in the same plane; and $\therefore$ BCA is a plane $\triangle$, and $BA=BA'''$, $CA=CA'''$, and BC is common to the two $\triangle$s BCA, BCA''', and so $\triangle BCA'''$ may be made to have its sides respectively coinciding with the sides of $\triangle BCA$, that is, BA with BA''', and CA with CA''', and as these sides are equal, A''' will coincide with A. Hence four equal and similar acute-angled triangles can be made into a tetrahedron.

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